Chapter 2

THE MATHEMATICS OF OPTIMIZATION

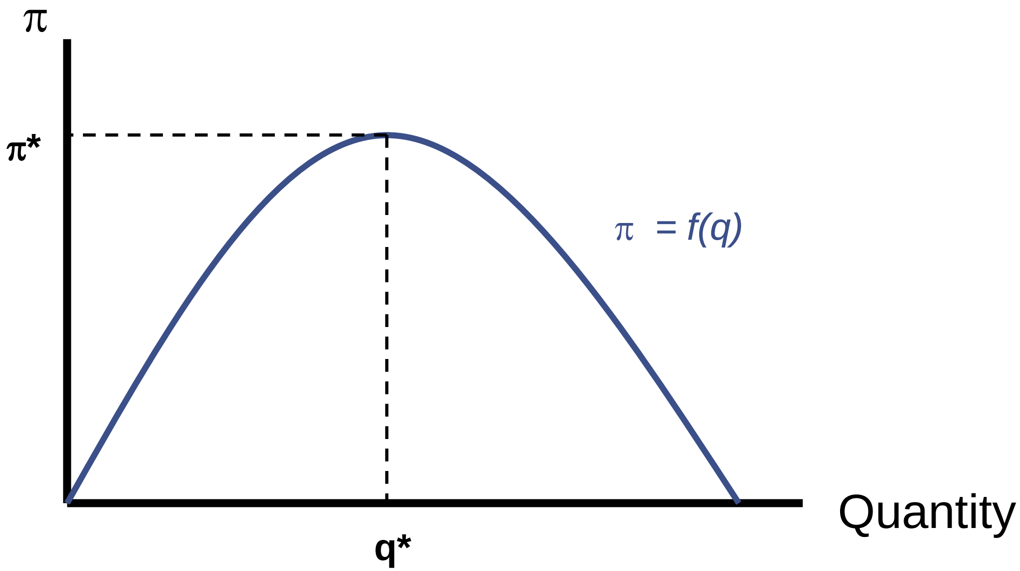
The Mathematics of Optimization

- Many economic theories begin with the assumption that an economic agent is seeking to find the optimal value of some function
  - consumers seek to maximize utility
  - firms seek to maximize profit
- This chapter introduces the mathematics common to these problems

Maximization of a Function of One Variable

- Simple example: Manager of a firm wishes to maximize profits

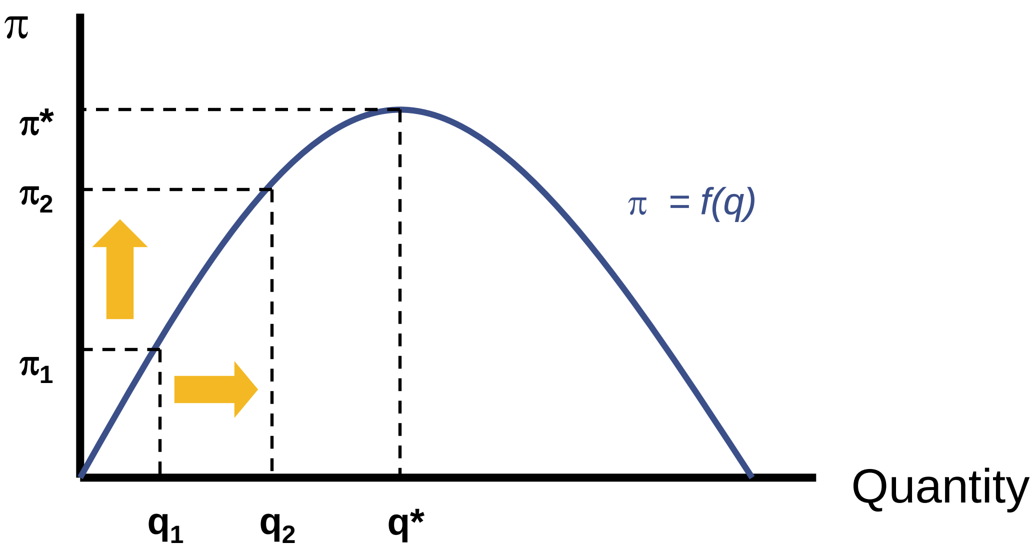
$$\pi = f(q)$$



Maximum profits of **π*** occur at q*

Maximization of a Function of One Variable

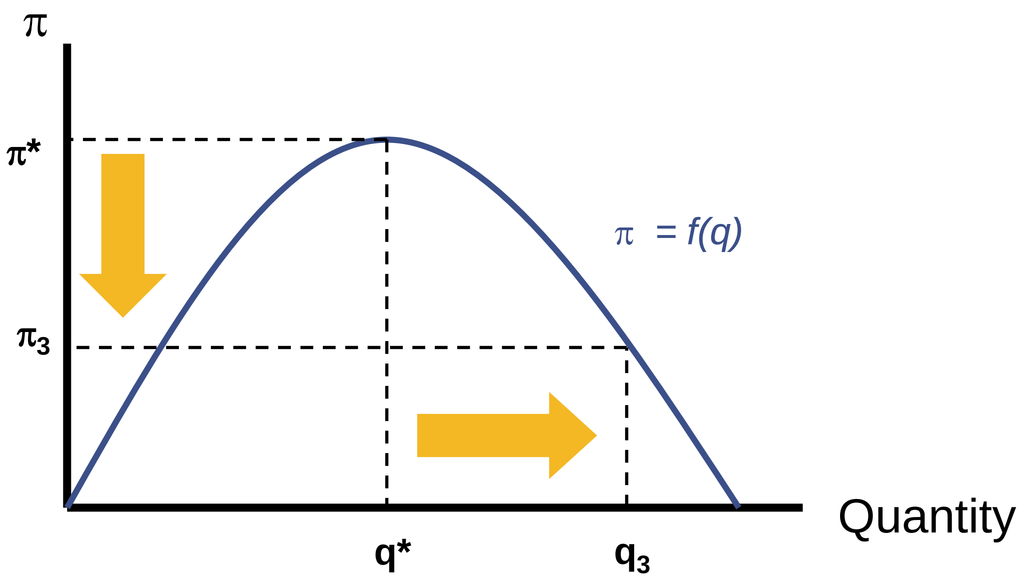
- The manager will likely try to vary q to see where the maximum profit occurs
  - an increase from q_1 to q_2 leads to a rise in π



$$\frac{\Delta\pi}{\Delta q} > 0$$

Maximization of a Function of One Variable

- If output is increased beyond q*, profit will decline
  - – an increase from q^* to q_3 leads to a drop in π



$$\frac{\Delta\pi}{\Delta q} < 0$$

Derivatives(导数)

- The derivative of $\pi = f(q)$ is the limit of $\Delta\pi/\Delta q$ for very small changes in q

$$\frac{d\pi}{dq} = \frac{df}{dq} = \lim_{h \to 0} \frac{f(q_1 + h) - f(q_1)}{h}$$

- The value of this ratio depends on the value of q_1

Value of a Derivative at a Point

- The evaluation of the derivative at the point $q = q_1$ can be denoted

$$\left.\frac{d\pi}{dq}\right|_{q=q_1}$$

- In our previous example,

$$\left.\frac{d\pi}{dq}\right|_{q=q_1} > 0 \qquad \left.\frac{d\pi}{dq}\right|_{q=q_3} < 0 \qquad \left.\frac{d\pi}{dq}\right|_{q=q^*} = 0$$

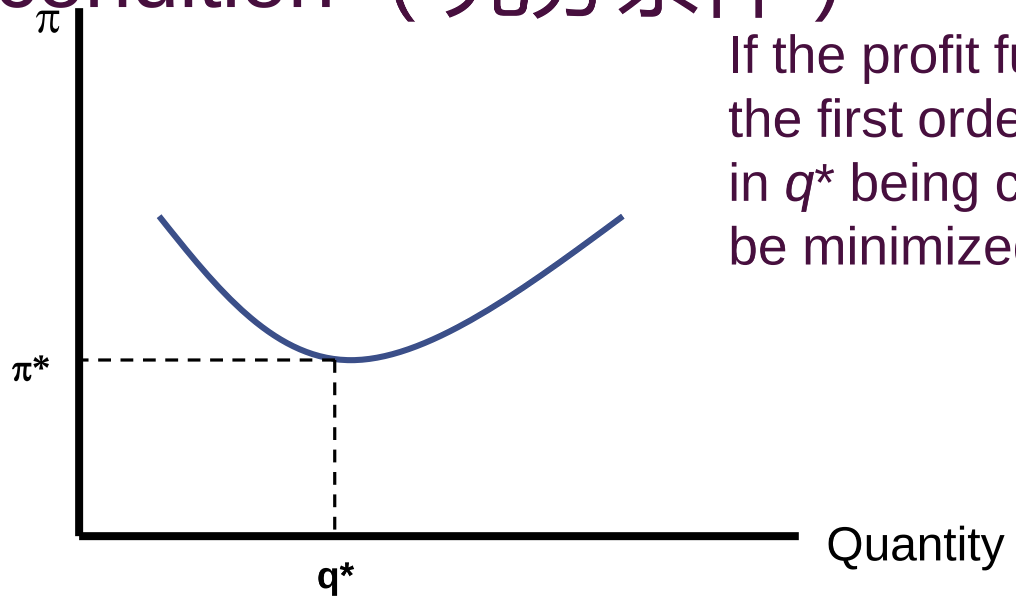
First Order Condition（一阶条件）for a Maximum

- For a function of one variable to attain its maximum value at some point, the derivative at that point must be zero

$$\left.\frac{df}{dq}\right|_{q=q^*} = 0$$

Second Order Conditions

- The first order condition ($d\pi/dq$) is a <u>necessary</u> condition（必要条件） for a maximum, but it is not a <u>sufficient</u> condition （充分条件）



If the profit function was u-shaped, the first order condition would result in *q** being chosen and π would be minimized

Second Order Conditions

- This must mean that, in order for q^* to be the optimum,

$$\frac{d\pi}{dq} > 0 \text{ for } q < q^* \quad \text{and} \quad \frac{d\pi}{dq} < 0 \text{ for } q > q^*$$

- Therefore, at q^*, $d\pi/dq$ must be decreasing

Second Derivatives

- The derivative of a derivative is called a second derivative（二阶导数）
- The second derivative can be denoted by

$$\frac{d^2\pi}{dq^2} \text{ or } \frac{d^2 f}{dq^2} \text{ or } f''(q)$$

Second Order Condition

- The second order condition to represent a (local) maximum is

$$\left.\frac{d^2\pi}{dq^2}\right|_{q=q*} = f''(q)\Big|_{q=q*} < 0$$

Rules for Finding Derivatives

1. If b is a constant, then $\frac{db}{dx} = 0$

2. If b is a constant, then $\frac{d[bf(x)]}{dx} = bf'(x)$

3. If b is constant, then $\frac{dx^b}{dx} = bx^{b-1}$

4. $\frac{d\ln x}{dx} = \frac{1}{x}$

Rules for Finding Derivatives

5. $\frac{da^x}{dx} = a^x \ln a$ for any constant a

– a special case of this rule is $de^x/dx = e^x$

Rules for Finding Derivatives

- Suppose that *f*(*x*) and *g*(*x*) are two functions of *x* and *f'*(*x*) and *g'*(*x*) exist
- Then

6. $$\frac{d[f(x)+g(x)]}{dx} = f'(x) + g'(x)$$

7. $$\frac{d[f(x)\cdot g(x)]}{dx} = f(x)g'(x) + f'(x)g(x)$$

Rules for Finding Derivatives

$$8.\ \frac{d\left(\dfrac{f(x)}{g(x)}\right)}{dx} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

provided that $g(x) \neq 0$

Rules for Finding Derivatives

- If $y = f(x)$ and $x = g(z)$ and if both $f'(x)$ and $g'(x)$ exist, then:

9. $$\frac{dy}{dz} = \frac{dy}{dx} \cdot \frac{dx}{dz} = \frac{df}{dx} \cdot \frac{dg}{dz}$$

- This is called the chain rule（链式法则）. The chain rule allows us to study how one variable (z) affects another variable (y) through its influence on some intermediate variable (x)

Rules for Finding Derivatives

- Some examples of the chain rule include

$$10.\quad \frac{de^{ax}}{dx} = \frac{de^{ax}}{d(ax)} \cdot \frac{d(ax)}{dx} = e^{ax} \cdot a = ae^{ax}$$

$$11.\quad \frac{d[\ln(ax)]}{dx} = \frac{d[\ln(ax)]}{d(ax)} \cdot \frac{d(ax)}{dx} = \ln(ax) \cdot a = a\ln(ax)$$

$$12.\quad \frac{d[\ln(x^2)]}{dx} = \frac{d[\ln(x^2)]}{d(x^2)} \cdot \frac{d(x^2)}{dx} = \frac{1}{x^2} \cdot 2x = \frac{2}{x}$$

Example of Profit Maximization

- Suppose that the relationship between profit and output is

$$\pi = 1{,}000q - 5q^2$$

- The first order condition for a maximum is

$$d\pi/dq = 1{,}000 - 10q = 0$$

$$q^* = 100$$

- Since the second derivative is always -10, $q = 100$ is a global maximum

Functions of Several Variables

- Most goals of economic agents depend on several variables
  - trade-offs（权衡，两相取舍） must be made
- The dependence of one variable (y) on a series of other variables ($x_1, x_2, \ldots, x_n$) is denoted by

$$y = f(x_1, x_2, \ldots, x_n)$$

Partial Derivatives（偏导数）

- The partial derivative of y with respect to x_1 is denoted by

$$\frac{\partial y}{\partial x_1} \text{ or } \frac{\partial f}{\partial x_1} \text{ or } f_{x_1} \text{ or } f_1$$

- It is understood that in calculating the partial derivative, all of the other x's are held constant

Partial Derivatives

- A more formal definition of the partial derivative is

$$\left.\frac{\partial f}{\partial x_1}\right|_{\bar{x}_2,\ldots,\bar{x}_n} = \lim_{h \to 0} \frac{f(x_1 + h, \bar{x}_2, \ldots, \bar{x}_n) - f(x_1, \bar{x}_2, \ldots, \bar{x}_n)}{h}$$

Calculating Partial Derivatives

1. If $y = f(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2$, then

$$\frac{\partial f}{\partial x_1} = f_1 = 2ax_1 + bx_2 \quad \text{and}$$

$$\frac{\partial f}{\partial x_2} = f_2 = bx_1 + 2cx_2$$

2. If $y = f(x_1, x_2) = e^{ax_1 + bx_2}$, then

$$\frac{\partial f}{\partial x_1} = f_1 = ae^{ax_1 + bx_2} \quad \text{and} \quad \frac{\partial f}{\partial x_2} = f_2 = be^{ax_1 + bx_2}$$

Calculating Partial Derivatives

3. If $y = f(x_1, x_2) = a\ln x_1 + b\ln x_2$, then

$$\frac{\partial f}{\partial x_1} = f_1 = \frac{a}{x_1} \text{ and } \frac{\partial f}{\partial x_2} = f_2 = \frac{b}{x_2}$$

Partial Derivatives

- Partial derivatives are the mathematical expression of the *ceteris paribus* assumption
  - show how changes in one variable affect some outcome when other influences are held constant

Partial Derivatives

- We must be concerned with how variables are measured
  - if q represents the quantity of gasoline demanded (measured in billions of gallons) and p represents the price in dollars per gallon, then $\partial q/\partial p$ will measure the change in demand (in billiions of gallons per year) for a dollar per gallon change in price

Elasticity（弹性）

- Elasticities measure the proportional effect of a change in one variable on another
  - unit free（与单位无关）
- The elasticity of *y* with respect to *x* is

$$e_{y,x} = \frac{\frac{\Delta y}{y}}{\frac{\Delta x}{x}} = \frac{\Delta y}{\Delta x} \cdot \frac{x}{y} = \frac{\partial y}{\partial x} \cdot \frac{x}{y}$$

Elasticity and Functional Form

- Suppose that

$$y = a + bx + \text{other terms}$$

- In this case,

$$e_{y,x} = \frac{\partial y}{\partial x} \cdot \frac{x}{y} = b \cdot \frac{x}{y} = b \cdot \frac{x}{a + bx + \cdots}$$

- $e_{y,x}$ is not constant
  - it is important to note the point at which the elasticity is to be computed

Elasticity and Functional Form

- Suppose that

$$y = ax^b$$

- In this case,

$$e_{y,x} = \frac{\partial y}{\partial x} \cdot \frac{x}{y} = abx^{b-1} \cdot \frac{x}{ax^b} = b$$

Elasticity and Functional Form

- Suppose that

$$\ln y = \ln a + b \ln x$$

- In this case,

$$e_{y,x} = \frac{\partial y}{\partial x} \cdot \frac{x}{y} = b \cdot \frac{\partial \ln y}{\partial \ln x}$$

- Elasticities can be calculated through logarithmic differentiation

Second-Order Partial Derivatives

- The partial derivative of a partial derivative is called a second-order partial derivative

$$\frac{\partial(\partial f / \partial x_i)}{\partial x_j} = \frac{\partial^2 f}{\partial x_j x_i} = f_{ij}$$

Young's Theorem

- Under general conditions, the order in which partial differentiation is conducted to evaluate second-order partial derivatives does not matter

$$f_{ij} = f_{ji}$$

Use of Second-Order Partials

- Second-order partials play an important role in many economic theories
- One of the most important is a variable's own second-order partial, f_{ii}
  - shows how the marginal influence of x_i on $y(\partial y/\partial x_i)$ changes as the value of x_i increases
  - a value of $f_{ii} < 0$ indicates diminishing marginal effectiveness

Total Differential

- Suppose that $y = f(x_1, x_2, \ldots, x_n)$
- If all x's are varied by a small amount, the total effect on y will be

$$dy = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \ldots + \frac{\partial f}{\partial x_n} dx_n$$

$$dy = f_1 dx_1 + f_2 dx_2 + \ldots + f_n dx_n$$

First-Order Condition for a Maximum (or Minimum)

- A necessary condition for a maximum (or minimum) of the function $f(x_1,x_2,\ldots,x_n)$ is that $dy = 0$ for any combination of small changes in the *x*'s
- The only way for this to be true is if

$$f_1 = f_2 = \ldots = f_n = 0$$

- A point where this condition holds is called a <u>critical point（奇点，零点）</u>

Finding a Maximum

- Suppose that y is a function of x_1 and x_2

$$y = -(x_1 - 1)^2 - (x_2 - 2)^2 + 10$$

$$y = -x_1^2 + 2x_1 - x_2^2 + 4x_2 + 5$$

- First-order conditions imply that

$$\frac{\partial y}{\partial x_1} = -2x_1 + 2 = 0$$

$$\frac{\partial y}{\partial x_2} = -2x_2 + 4 = 0$$

OR

$$x_1^* = 1$$

$$x_2^* = 2$$

Production Possibility Frontier

- Earlier example: $2x^2 + y^2 = 225$
- Can be rewritten: $f(x,y) = 2x^2 + y^2 - 225 = 0$
- Because $f_x = 4x$ and $f_y = 2y$, the opportunity cost trade-off between x and y is

$$\frac{dy}{dx} = \frac{-f_x}{f_y} = \frac{-4x}{2y} = \frac{-2x}{y}$$

Implicit Function Theorem（隐函数定理）

- It may not always be possible to solve implicit functions of the form $g(x,y)=0$ for unique explicit functions of the form $y = f(x)$
  - mathematicians have derived the necessary conditions
  - in many economic applications, these conditions are the same as the second-order conditions for a maximum (or minimum)

The Envelope Theorem（包络定理）

- The envelope theorem concerns how the optimal value for a particular function changes when a parameter of the function changes
- This is easiest to see by using an example

The Envelope Theorem

- Suppose that *y* is a function of *x*

$$y = -x^2 + ax$$

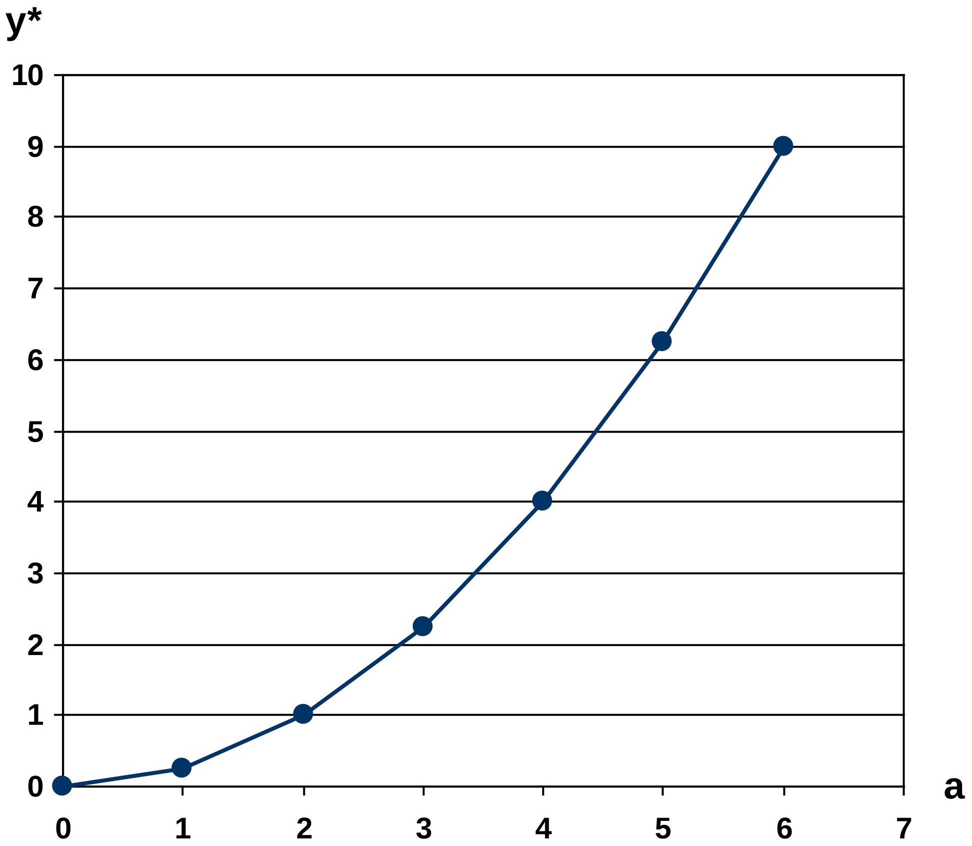
- For different values of *a*, this function represents a family of inverted parabolas（翻转抛物线）
- If *a* is assigned a specific value, then *y* becomes a function of *x* only and the value of *x* that maximizes *y* can be calculated

The Envelope Theorem

Optimal Values of *x* and *y* for alternative values of *a*

Value of a	Value of x^*	Value of y^*
0	0	0
1	1/2	1/4
2	1	1
3	3/2	9/4
4	2	4
5	5/2	25/4
6	3	9

The Envelope Theorem



As *a* increases, the maximal value for *y* (*y**) increases

The relationship between *a* and *y* is quadratic

The Envelope Theorem

- Suppose we are interested in how y^* changes as a changes
- There are two ways we can do this
  - calculate the slope of y directly
  - hold x constant at its optimal value and calculate $\partial y/\partial a$ directly

The Envelope Theorem

- To calculate the slope of the function, we must solve for the optimal value of *x* for any value of *a*

$$dy/dx = -2x + a = 0$$

$$x^* = a/2$$

- Substituting, we get

$$y^* = -(x^*)^2 + a(x^*) = -(a/2)^2 + a(a/2)$$

$$y^* = -a^2/4 + a^2/2 = a^2/4$$

The Envelope Theorem

- Therefore,

$$dy^*/da = 2a/4 = a/2 = x^*$$

- But, we can save time by using the envelope theorem
  - for small changes in *a*, dy^*/da can be computed by holding *x* at x^* and calculating $\partial y/ \partial a$ directly from *y*

The Envelope Theorem

$$\partial y / \partial a = x$$

- Holding $x = x^*$

$$\partial y / \partial a = x^* = a/2$$

- This is the same result found earlier

The Envelope Theorem

- The envelope theorem states that the change in the optimal value of a function with respect to a parameter of that function can be found by partially differentiating the objective function while holding *x* (or several *x*'s) at its optimal value

$$\frac{dy^*}{da} = \frac{\partial y}{\partial a}\{x = x^*(a)\}$$

The Envelope Theorem

- The envelope theorem can be extended to the case where y is a function of several variables

$$y = f(x_1,\ldots x_n,a)$$

- Finding an optimal value for *y* would consist of solving *n* first-order equations

$$\partial y/\partial x_i = 0 \quad (i = 1,\ldots,n)$$

The Envelope Theorem

- Optimal values for theses *x*'s would be determined that are a function of *a*

$$x_1^* = x_1^*(a)$$

$$x_2^* = x_2^*(a)$$

$$\vdots$$

$$x_n^* = x_n^*(a)$$

The Envelope Theorem

- Substituting into the original objective function yields an expression for the optimal value of *y* (*y**)

$$y^* = f\,[x_1^*(a), x_2^*(a), \ldots, x_n^*(a), a]$$

- Differentiating yields

$$\frac{dy^*}{da} = \frac{\partial f}{\partial x_1} \cdot \frac{dx_1}{da} + \frac{\partial f}{\partial x_2} \cdot \frac{dx_2}{da} + \ldots + \frac{\partial f}{\partial x_n} \cdot \frac{dx_n}{da} + \frac{\partial f}{\partial a}$$

The Envelope Theorem

- Because of first-order conditions, all terms except $\partial f/\partial a$ are equal to zero if the x's are at their optimal values

- Therefore,

$$\frac{dy^*}{da} = \frac{\partial f}{\partial a}\{x = x^*(a)\}$$

Constrained Maximization

- What if not all values for the *x*'s are feasible?
  - the values of *x* may all have to be positive
  - a consumer's choices are limited by the amount of purchasing power available
- One method used to solve constrained maximization problems is the <u>Lagrangian multiplier method</u>

Lagrangian Multiplier Method

- Suppose that we wish to find the values of $x_1, x_2, \ldots, x_n$ that maximize

$$y = f(x_1, x_2, \ldots, x_n)$$

subject to a constraint that permits only certain values of the x's to be used

$$g(x_1, x_2, \ldots, x_n) = 0$$

Lagrangian Multiplier Method

- The Lagrangian multiplier method starts with setting up the expression

  $$\mathbf{L} = f(x_1, x_2, \ldots, x_n) + \lambda g(x_1, x_2, \ldots, x_n)$$

  where λ is an additional variable called a Lagrangian multiplier
- When the constraint holds, $\mathbf{L} = f$ because $g(x_1, x_2, \ldots, x_n) = 0$

Lagrangian Multiplier Method

- First-Order Conditions

$$\partial \mathbf{L}/\partial x_1 = f_1 + \lambda g_1 = 0$$

$$\partial \mathbf{L}/\partial x_2 = f_2 + \lambda g_2 = 0$$

$$\vdots$$

$$\partial \mathbf{L}/\partial x_n = f_n + \lambda g_n = 0$$

$$\partial \mathbf{L}/\partial \lambda = g(x_1, x_2, \ldots, x_n) = 0$$

Lagrangian Multiplier Method

- The first-order conditions can generally be solved for x_1, x_2,…, x_n and λ
- The solution will have two properties:
  - the *x*'s will obey the constraint
  - these *x*'s will make the value of **L** (and therefore *f*) as large as possible

Lagrangian Multiplier Method

- The Lagrangian multiplier (λ) has an important economic interpretation
- The first-order conditions imply that

$$f_1/\text{-}g_1 = f_2/\text{-}g_2 = \ldots = f_n/\text{-}g_n = \lambda$$

  - the numerators above measure the marginal benefit that one more unit of x_i will have for the function f
  - the denominators reflect the added burden on the constraint of using more x_i

Lagrangian Multiplier Method

- At the optimal choices for the *x*'s, the ratio of the marginal benefit of increasing x_i to the marginal cost of increasing x_i should be the same for every *x*
- λ is the common cost-benefit ratio for all of the x's

$$\lambda = \frac{\text{marginal benefit of } x_i}{\text{marginal cost of } x_i}$$

Lagrangian Multiplier Method

- If the constraint was relaxed slightly, it would not matter which *x* is changed
- The Lagrangian multiplier provides a measure of how the relaxation in the constraint will affect the value of *y*
- λ provides a “shadow price” (影子价格 ） to the constraint

Lagrangian Multiplier Method

- A high value of λ indicates that *y* could be increased substantially by relaxing the constraint
  - each *x* has a high cost-benefit ratio
- A low value of λ indicates that there is not much to be gained by relaxing the constraint
- λ=0 implies that the constraint is not binding

Duality （对偶性）

- Any constrained maximization problem has associated with it a dual problem in constrained minimization that focuses attention on the constraints in the original problem

Duality

- Individuals maximize utility subject to a budget constraint
  - dual problem: individuals minimize the expenditure needed to achieve a given level of utility
- Firms minimize the cost of inputs to produce a given level of output
  - dual problem: firms maximize output for a given cost of inputs purchased

Constrained Maximization

- Suppose a farmer had a certain length of fence (P) and wished to enclose the largest possible rectangular shape
- Let x be the length of one side
- Let y be the length of the other side
- Problem: choose x and y so as to maximize the area ($A = x \cdot y$) subject to the constraint that the perimeter is fixed at $P = 2x + 2y$

Constrained Maximization

- Setting up the Lagrangian multiplier

$$\mathbf{L} = x \cdot y + \lambda(P - 2x - 2y)$$

- The first-order conditions for a maximum are

$$\partial \mathbf{L}/\partial x = y - 2\lambda = 0$$

$$\partial \mathbf{L}/\partial y = x - 2\lambda = 0$$

$$\partial \mathbf{L}/\partial \lambda = P - 2x - 2y = 0$$

Constrained Maximization

- Since $y/2 = x/2 = \lambda$, x must be equal to y
  - the field should be square
  - x and y should be chosen so that the ratio of marginal benefits to marginal costs should be the same
- Since $x = y$ and $y = 2\lambda$, we can use the constraint to show that

$$x = y = P/4$$

$$\lambda = P/8$$

Constrained Maximization

- Interpretation of the Lagrangian multiplier
  - if the farmer was interested in knowing how much more field could be fenced by adding an extra yard of fence, λ suggests that he could find out by dividing the present perimeter (P) by 8
  - thus, the Lagrangian multiplier provides information about the implicit value of the constraint

Constrained Maximization

- Dual problem: choose *x* and *y* to minimize the amount of fence required to surround the field

$$\text{minimize } P = 2x + 2y \text{ subject to } A = x \cdot y$$

- Setting up the Lagrangian:

$$\mathbf{L}^D = 2x + 2y + \lambda^D(A - x \cdot y)$$

Constrained Maximization

- First-order conditions:

$$\partial \mathbf{L}^D / \partial x = 2 - \lambda^D \cdot y = 0$$

$$\partial \mathbf{L}^D / \partial y = 2 - \lambda^D \cdot x = 0$$

$$\partial \mathbf{L}^D / \partial \lambda^D = A - x \cdot y = 0$$

- Solving, we get

$$x = y = A^{1/2}$$

- The Lagrangian multiplier (λ^D) = $2A^{-1/2}$

Envelope Theorem & Constrained Maximization

- Suppose that we want to maximize

$$y = f(x_1,\ldots,x_n;a)$$

subject to the constraint

$$g(x_1,\ldots,x_n;a) = 0$$

- One way to solve would be to set up the Lagrangian expression and solve the first-order conditions

Envelope Theorem & Constrained Maximization

- Alternatively, it can be shown that

$$dy^*/da = \partial \mathbf{L}/\partial a(x_1^*,\ldots,x_n^*;a)$$

- The change in the maximal value of *y* that results when *a* changes can be found by partially differentiating **L** and evaluating the partial derivative at the optimal point

Inequality Constraints

- In some economic problems the constraints need not hold exactly
- For example, suppose we seek to maximize $y = f(x_1,x_2)$ subject to

$$g(x_1,x_2) \geq 0,$$

$$x_1 \geq 0, \text{ and}$$

$$x_2 \geq 0$$

Inequality Constraints

- One way to solve this problem is to introduce three new variables (a, b, and c) that convert the inequalities into equalities
- To ensure that the inequalities continue to hold, we will square these new variables to ensure that their values are positive

Inequality Constraints

$$g(x_1,x_2) - a^2 = 0;$$

$$x_1 - b^2 = 0; \text{ and}$$

$$x_2 - c^2 = 0$$

- Any solution that obeys these three equality constraints will also obey the inequality constraints

Inequality Constraints

- We can set up the Lagrangian

$$\mathbf{L} = f(x_1,x_2) + \lambda_1[g(x_1,x_2) - a^2] + \lambda_2[x_1 - b^2] + \lambda_3[x_2 - c^2]$$

- This will lead to eight first-order conditions

Inequality Constraints

$$\partial \mathbf{L}/\partial x_1 = f_1 + \lambda_1 g_1 + \lambda_2 = 0$$

$$\partial \mathbf{L}/\partial x_2 = f_1 + \lambda_1 g_2 + \lambda_3 = 0$$

$$\partial \mathbf{L}/\partial a = -2a\lambda_1 = 0$$

$$\partial \mathbf{L}/\partial b = -2b\lambda_2 = 0$$

$$\partial \mathbf{L}/\partial c = -2c\lambda_3 = 0$$

$$\partial \mathbf{L}/\partial \lambda_1 = g(x_1,x_2) - a^2 = 0$$

$$\partial \mathbf{L}/\partial \lambda_2 = x_1 - b^2 = 0$$

$$\partial \mathbf{L}/\partial \lambda_3 = x_2 - c^2 = 0$$

Inequality Constraints

- According to the third condition, either a or $\lambda_1 = 0$
  - if $a = 0$, the constraint $g(x_1,x_2)$ holds exactly
  - if $\lambda_1 = 0$, the availability of some slackness of the constraint implies that its value to the objective function is 0
- Similar complemetary slackness（互补松弛）relationships also hold for x_1 and x_2

$\partial\lambda_1\ \partial \mathbf{L}/\partial\lambda_1 = 0$

Inequality Constraints

- These results are sometimes called Kuhn-Tucker conditions
  - they show that solutions to optimization problems involving inequality constraints will differ from similar problems involving equality constraints in rather simple ways
  - we cannot go wrong by working primarily with constraints involving equalities

Second Order Conditions - Functions of One Variable

- Let $y = f(x)$
- A necessary condition for a maximum is that

$$dy/dx = f'(x) = 0$$

- To ensure that the point is a maximum, y must be decreasing for movements away from it

Second Order Conditions - Functions of One Variable

- The total differential measures the change in y

$$dy = f'(x)\,dx$$

- To be at a maximum, dy must be decreasing for small increases in x
- To see the changes in dy, we must use the second derivative of y

Second Order Conditions - Functions of One Variable

$$d^2y = \frac{d[f'(x)dx]}{dx} \cdot dx = f''(x)dx \cdot dx = f''(x)dx^2$$

- Note that $d^2y < 0$ implies that $f''(x)dx^2 < 0$
- Since dx^2 must be positive, $f''(x) < 0$

- This means that the function f must have a concave (凹) shape at the critical point

Second Order Conditions - Functions of Two Variables

- Suppose that $y = f(x_1, x_2)$
- First order conditions for a maximum are

$$\partial y/\partial x_1 = f_1 = 0$$

$$\partial y/\partial x_2 = f_2 = 0$$

- To ensure that the point is a maximum, *y* must diminish for movements in any direction away from the critical point

Second Order Conditions - Functions of Two Variables

- The slope in the x_1 direction (f_1) must be diminishing at the critical point
- The slope in the x_2 direction (f_2) must be diminishing at the critical point
- But, conditions must also be placed on the cross-partial derivative ($f_{12} = f_{21}$) to ensure that dy is decreasing for all movements through the critical point

Second Order Conditions - Functions of Two Variables

- The total differential of *y* is given by

$$dy = f_1\, dx_1 + f_2\, dx_2$$

- The differential of that function is

$$d^2y = (f_{11}dx_1 + f_{12}dx_2)dx_1 + (f_{21}dx_1 + f_{22}dx_2)dx_2$$

$$d^2y = f_{11}dx_1^2 + f_{12}dx_2dx_1 + f_{21}dx_1\, dx_2 + f_{22}dx_2^2$$

- By Young's theorem, $f_{12} = f_{21}$ and

$$d^2y = f_{11}dx_1^2 + 2f_{12}dx_1dx_2 + f_{22}dx_2^2$$

Second Order Conditions - Functions of Two Variables

$$d^2y = f_{11}dx_1^2 + 2f_{12}dx_1dx_2 + f_{22}dx_2^2$$

- For this equation to be unambiguously negative for any change in the x's, f_{11} and f_{22} must be negative
- If $dx_2 = 0$, then $d^2y = f_{11}\,dx_1^2$
  - for $d^2y < 0$, $f_{11} < 0$
- If $dx_1 = 0$, then $d^2y = f_{22}\,dx_2^2$
  - for $d^2y < 0$, $f_{22} < 0$

Second Order Conditions - Functions of Two Variables

$$d^2y = f_{11}dx_1^2 + 2f_{12}dx_1dx_2 + f_{22}dx_2^2$$

- If neither dx_1 nor dx_2 is zero, then d^2y will be unambiguously negative only if

$$f_{11} f_{22} - f_{12}^2 > 0$$

  – the second partial derivatives (f_{11} and f_{22}) must be sufficiently negative so that they outweigh any possible perverse effects from the cross-partial derivatives ($f_{12} = f_{21}$)

Constrained Maximization

- Suppose we want to choose x_1 and x_2 to maximize

$$y = f(x_1, x_2)$$

- subject to the linear constraint

$$c - b_1x_1 - b_2x_2 = 0$$

- We can set up the Lagrangian

$$\mathbf{L} = f(x_1, x_2) + \lambda(c - b_1x_1 - b_2x_2)$$

Constrained Maximization

- The first-order conditions are

$$f_1 - \lambda b_1 = 0$$

$$f_2 - \lambda b_2 = 0$$

$$c - b_1x_1 - b_2x_2 = 0$$

- To ensure we have a maximum, we must use the "second" total differential

$$d^2y = f_{11}dx_1^2 + 2f_{12}dx_1dx_2 + f_{22}dx_2^2$$

Constrained Maximization

- Only the values of x_1 and x_2 that satisfy the constraint can be considered valid alternatives to the critical point
- Thus, we must calculate the total differential of the constraint

$$-b_1\, dx_1 - b_2\, dx_2 = 0$$

$$dx_2 = -(b_1/b_2)dx_1$$

- These are the allowable relative changes in x_1 and x_2

Constrained Maximization

- Because the first-order conditions imply that $f_1/f_2 = b_1/b_2$, we can substitute and get

$$dx_2 = -(f_1/f_2)\, dx_1$$

- Since

$$d^2y = f_{11}dx_1^2 + 2f_{12}dx_1dx_2 + f_{22}dx_2^2$$

we can substitute for dx_2 and get

$$d^2y = f_{11}dx_1^2 - 2f_{12}(f_1/f_2)dx_1^2 + f_{22}(f_1^2/f_2^2)dx_1^2$$

Constrained Maximization

- Combining terms and rearranging

$$d^2y = f_{11} f_2^2 - 2f_{12}f_1f_2 + f_{22}f_1^2 [dx_1^2/ f_2^2]$$

- Therefore, for $d^2y < 0$, it must be true that

$$f_{11} f_2^2 - 2f_{12}f_1f_2 + f_{22}f_1^2 < 0$$

- This equation characterizes a set of functions termed quasi-concave（拟凹）functions
  – any two points within the set can be joined by a line contained completely in the set

Concave and Quasi-Concave Functions

- The differences between concave and quasi-concave functions can be illustrated with the function

$$y = f(x_1,x_2) = (x_1 \cdot x_2)^k$$

where the *x*'s take on only positive values and *k* can take on a variety of positive values

Concave and Quasi-Concave Functions

- No matter what value *k* takes, this function is quasi-concave
- Whether or not the function is concave depends on the value of *k*
  - if $k < 0.5$, the function is concave
  - if $k > 0.5$, the function is convex

Homogeneous（齐次）Functions

- A function $f(x_1,x_2,\ldots x_n)$ is said to be <u>homogeneous of degree k</u> if

$$f(tx_1,tx_2,\ldots tx_n) = t^k f(x_1,x_2,\ldots x_n)$$

  - when a function is homogeneous of degree one, a doubling of all of its arguments doubles the value of the function itself
  - when a function is homogeneous of degree zero, a doubling of all of its arguments leaves the value of the function unchanged

Homogeneous Functions

- If a function is homogeneous of degree k, the partial derivatives of the function will be homogeneous of degree k-1

Euler's Theorem

- If we differentiate the definition for homogeneity with respect to the proportionality factor *t*, we get

$$kt^{k-1}f(x_1,\ldots,x_n) = x_1f_1(tx_1,\ldots,tx_n) + \ldots + x_nf_n(x_1,\ldots,x_n)$$

- This relationship is called Euler's theorem

Euler's Theorem

- Euler's theorem shows that, for homogeneous functions, there is a definite relationship between the values of the function and the values of its partial derivatives

Homothetic（同位） Functions

- A homothetic function is one that is formed by taking a monotonic transformation of a homogeneous function
  - they do not possess the homogeneity properties of their underlying functions

Homothetic Functions

- For both homogeneous and homothetic functions, the implicit trade-offs among the variables in the function depend only on the ratios of those variables, not on their absolute values

Homothetic Functions

- Suppose we are examining the simple, two variable implicit function $f(x,y) = 0$
- The implicit trade-off between x and y for a two-variable function is

$$dy/dx = -f_x/f_y$$

- If we assume f is homogeneous of degree k, its partial derivatives will be homogeneous of degree k-1

Homothetic Functions

- The implicit trade-off between *x* and *y* is

$$\frac{dy}{dx} = -\frac{t^{k-1} f_x(tx,ty)}{t^{k-1} f_y(tx,ty)} = -\frac{f_x(tx,ty)}{f_y(tx,ty)}$$

- If $t = 1/y$,

$$\frac{dy}{dx} = -\frac{F' f_x\left(\frac{x}{y},1\right)}{F' f_y\left(\frac{x}{y},1\right)} = -\frac{f_x\left(\frac{x}{y},1\right)}{f_y\left(\frac{x}{y},1\right)}$$

Homothetic Functions

- The trade-off is unaffected by the monotonic transformation and remains a function only of the ratio x to y

Important Points to Note:

- Using mathematics provides a convenient, short-hand way for economists to develop their models
  - implications of various economic assumptions can be studied in a simplified setting through the use of such mathematical tools

Important Points to Note:

- Derivatives are often used in economics because economists are interested in how marginal changes in one variable affect another
  - partial derivatives incorporate the *ceteris paribus* assumption used in most economic models

Important Points to Note:

- The mathematics of optimization is an important tool for the development of models that assume that economic agents rationally pursue some goal
  - the first-order condition for a maximum requires that all partial derivatives equal zero

Important Points to Note:

- Most economic optimization problems involve constraints on the choices that agents can make
  - the first-order conditions for a maximum suggest that each activity be operated at a level at which the ratio of the marginal benefit of the activity to its marginal cost

Important Points to Note:

- The Lagrangian multiplier is used to help solve constrained maximization problems
  - the Lagrangian multiplier can be interpreted as the implicit value (shadow price) of the constraint

Important Points to Note:

- The implicit function theorem illustrates the dependence of the choices that result from an optimization problem on the parameters of that problem

Important Points to Note:

- The envelope theorem examines how optimal choices will change as the problem's parameters change
- Some optimization problems may involve constraints that are inequalities rather than equalities

Important Points to Note:

- First-order conditions are necessary but not sufficient for ensuring a maximum or minimum
  - second-order conditions that describe the curvature（曲度） of the function must be checked

Important Points to Note:

- Certain types of functions occur in many economic problems
  - quasi-concave functions obey the second-order conditions of constrained maximum or minimum problems when the constraints are linear
  - homothetic functions have the property that implicit trade-offs among the variables depend only on the ratios of these variables