# Chapter 4

## UTILITY MAXIMIZATION AND CHOICE

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# Complaints about the Economic Approach

- No real individuals make the kinds of "lightning calculations"(闪电计算) required for utility maximization
- But the utility-maximization model predicts many aspects of behavior
- Thus, economists assume that people behave *as if* they made such calculations

# Complaints about the Economic Approach

- The economic model of choice is extremely selfish because no one has solely self-centered goals
- But nothing in the utility-maximization model prevents individuals from deriving satisfaction from “doing good”

# Optimization Principle

- To maximize utility, given a fixed amount of income to spend, an individual will buy the goods and services:
  - that exhaust his or her total income
  - for which the psychic rate of trade-off between any goods (the *MRS*) is equal to the rate at which goods can be traded for one another in the marketplace

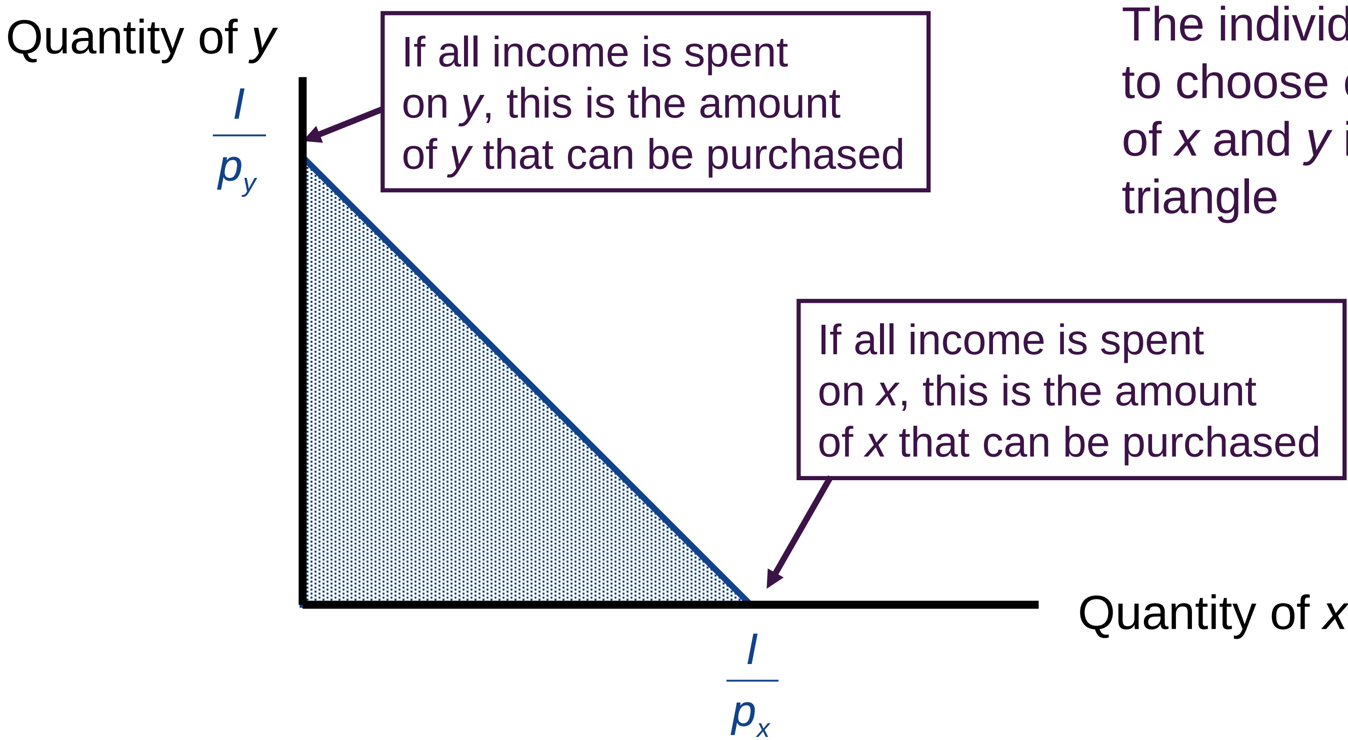
# A Numerical Illustration

- Assume that the individual's *MRS* = 1
  - willing to trade one unit of *x* for one unit of *y*
- Suppose the price of *x* = $2 and the price of *y* = $1
- The individual can be made better off
  - trade 1 unit of *x* for 2 units of *y* in the marketplace

# The Budget Constraint

- Assume that an individual has *I* dollars to allocate between good *x* and good *y*

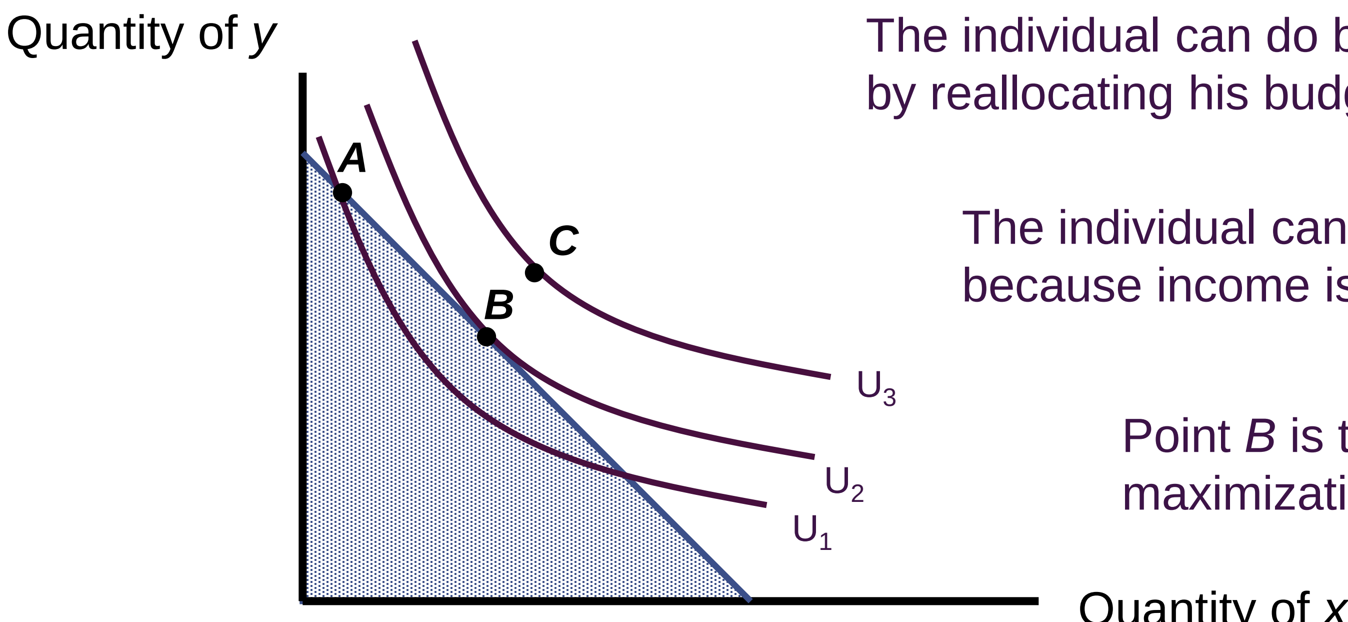
$$p_x x + p_y y \le I$$




The individual can afford to choose only combinations of *x* and *y* in the shaded triangle

# First-Order Conditions for a Maximum

- We can add the individual's utility map to show the utility-maximization process



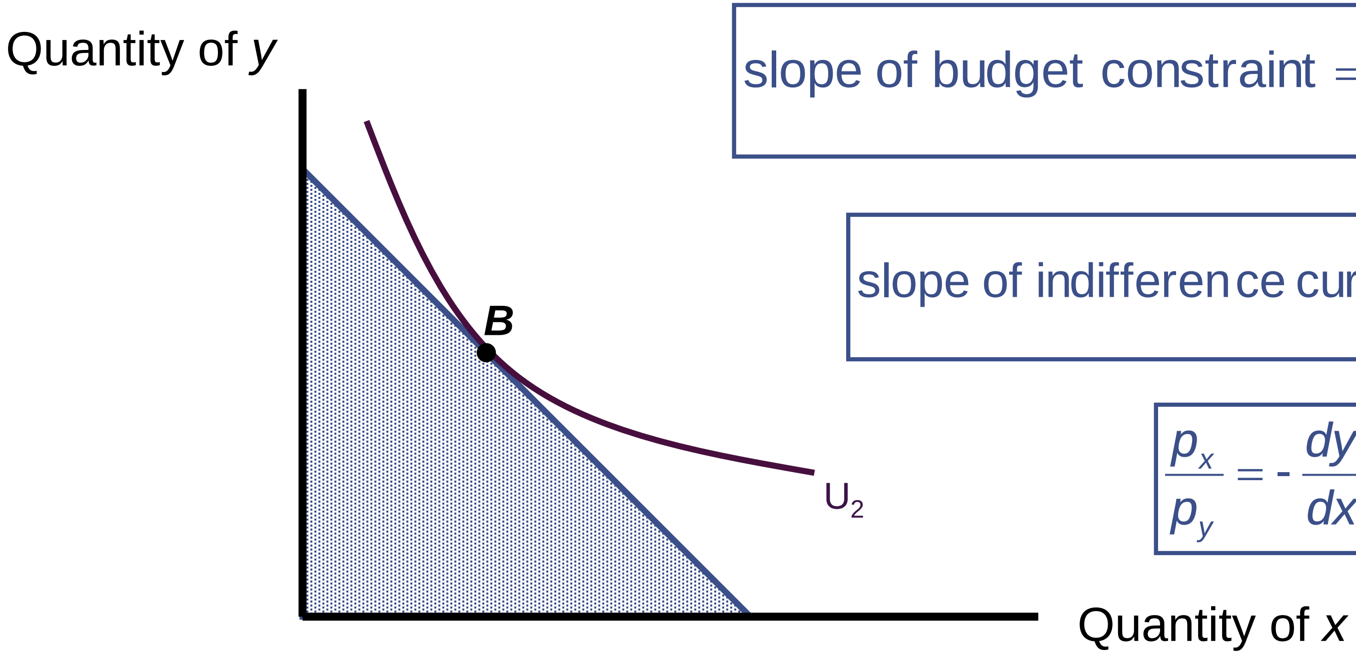

The individual can do better than point *A* by reallocating his budget

The individual cannot have point *C* because income is not large enough

Point *B* is the point of utility maximization

# First-Order Conditions for a Maximum

- Utility is maximized where the indifference curve is tangent to the budget constraint




$$\text{slope of budget constraint} = -\frac{p_x}{p_y}$$

$$\text{slope of indifference curve} = \left.\frac{dy}{dx}\right|_{U=\text{constant}}$$

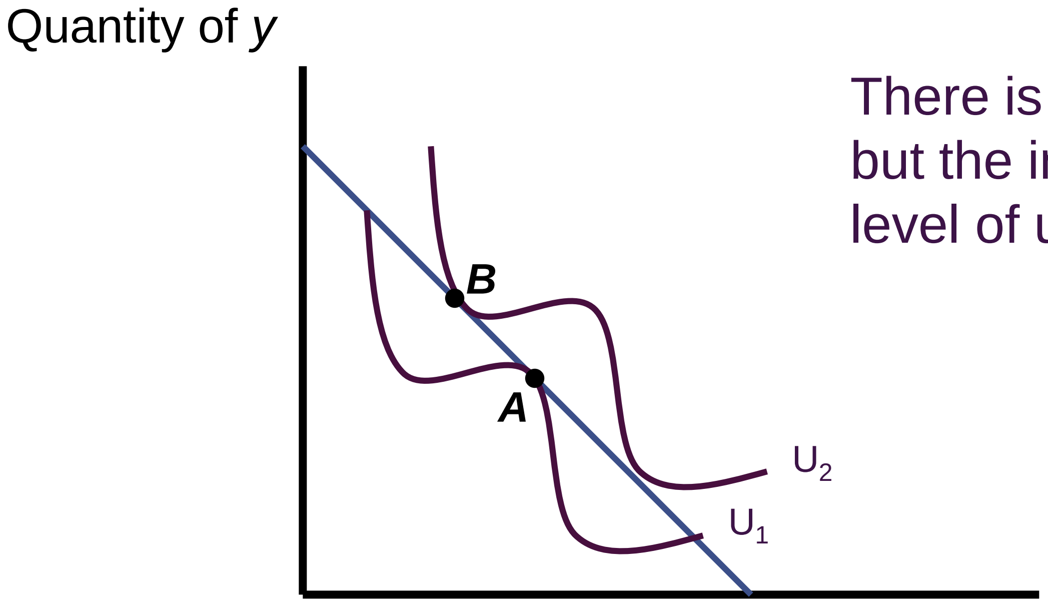
$$\frac{p_x}{p_y} = -\left.\frac{dy}{dx}\right|_{U=\text{constant}} = MRS$$

# Second-Order Conditions for a Maximum

- The tangency rule is only necessary but not sufficient unless we assume that *MRS* is diminishing
  - if *MRS* is diminishing, then indifference curves are strictly convex
- If *MRS* is not diminishing, then we must check second-order conditions to ensure that we are at a maximum

# Second-Order Conditions for a Maximum

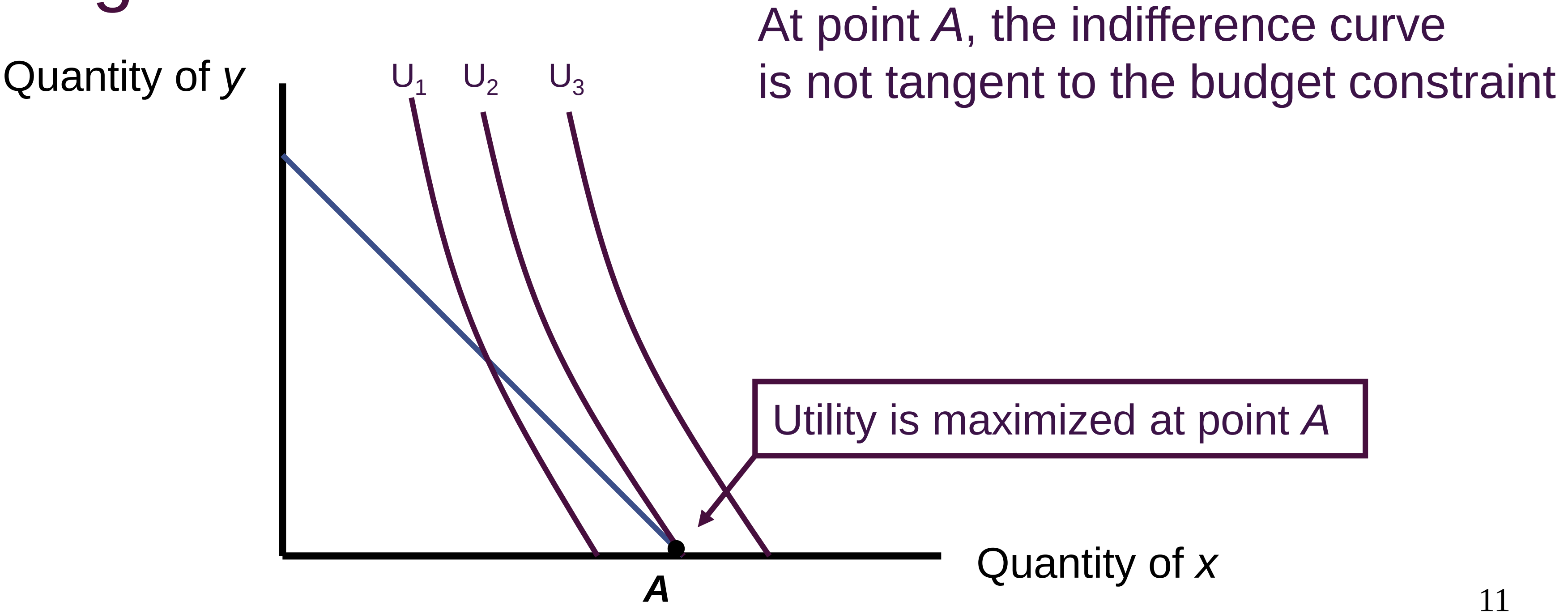
- The tangency rule is only a necessary condition
  - we need *MRS* to be diminishing




There is a tangency at point *A*, but the individual can reach a higher level of utility at point *B*

# Corner Solutions

- In some situations, individuals' preferences may be such that they can maximize utility by choosing to consume only one of the goods



# The （general） *n*-Good Case

- The individual's objective is to maximize

$$\text{utility} = U(x_1, x_2, \ldots, x_n)$$

subject to the budget constraint

$$I = p_1x_1 + p_2x_2 + \ldots + p_nx_n$$

- Set up the Lagrangian:

$$\mathbf{L} = U(x_1, x_2, \ldots, x_n) + \lambda(I - p_1x_1 - p_2x_2 - \ldots - p_nx_n)$$

# The *n*-Good Case

- First-order conditions for an **interior** maximum:

$$\partial \mathbf{L}/\partial x_1 = \partial U/\partial x_1 - \lambda p_1 = 0$$

$$\partial \mathbf{L}/\partial x_2 = \partial U/\partial x_2 - \lambda p_2 = 0$$

$$\vdots$$

$$\partial \mathbf{L}/\partial x_n = \partial U/\partial x_n - \lambda p_n = 0$$

$$\partial \mathbf{L}/\partial \lambda = I - p_1x_1 - p_2x_2 - \ldots - p_nx_n = 0$$

# Implications of First-Order Conditions

- For any two goods,

$$\frac{\partial U / \partial x_i}{\partial U / \partial x_j} = \frac{p_i}{p_j}$$

- This implies that at the optimal allocation of income

$$MRS\,(x_i \text{ for } x_j) = \frac{p_i}{p_j}$$

# Interpreting the Lagrangian Multiplier

$$\lambda = \frac{\partial U / \partial x_1}{p_1} = \frac{\partial U / \partial x_2}{p_2} = \ldots = \frac{\partial U / \partial x_n}{p_n}$$

$$\lambda = \frac{MU_{x_1}}{p_1} = \frac{MU_{x_2}}{p_2} = \ldots = \frac{MU_{x_n}}{p_n}$$

- $\lambda$ is the marginal utility of an extra dollar of consumption expenditure
  - the marginal utility of income

# Interpreting the Lagrangian Multiplier

- At the margin, the price of a good represents the consumer's evaluation of the utility of the last unit consumed
  – how much the consumer is willing to pay for the last unit

$$p_i = \frac{MU_{x_i}}{\lambda}$$

# Corner Solutions

- When **corner** solutions are involved, the first-order conditions must be modified:

$$\partial \mathbf{L}/\partial x_i = \partial U/\partial x_i - \lambda p_i \leq 0 \quad (i = 1,\ldots,n)$$

- If for some i, $\partial \mathbf{L}/\partial x_i = \partial U/\partial x_i - \lambda p_i < 0$, then $x_i = 0$
- This means that

$$p_i > \frac{\partial U / \partial x_i}{\lambda} = \frac{MU_{x_i}}{\lambda}$$

  – any good whose price exceeds its marginal value to the consumer will not be purchased

# Cobb-Douglas Demand Functions

- Cobb-Douglas utility function:

$$U(x,y) = x^{\alpha}y^{\beta}$$

- Setting up the Lagrangian:

$$\mathbf{L} = x^{\alpha}y^{\beta} + \lambda(I - p_x x - p_y y)$$

- First-order conditions:

$$\partial \mathbf{L}/\partial x = \alpha x^{\alpha-1}y^{\beta} - \lambda p_x = 0$$

$$\partial \mathbf{L}/\partial y = \beta x^{\alpha}y^{\beta-1} - \lambda p_y = 0$$

$$\partial \mathbf{L}/\partial \lambda = I - p_x x - p_y y = 0$$

# Cobb-Douglas Demand Functions

- First-order conditions imply:

$$\alpha y/\beta x = p_x/p_y$$

- Since $\alpha + \beta = 1$:

$$p_y y = (\beta/\alpha)p_x x = [(1- \alpha)/\alpha]p_x x$$

- Substituting into the budget constraint:

$$I = p_x x + [(1- \alpha)/\alpha]p_x x = (1/\alpha)p_x x$$

# Cobb-Douglas Demand Functions

- Solving for *x* yields

$$x^* = \frac{\alpha I}{p_x}$$

- Solving for *y* yields

$$y^* = \frac{\beta I}{p_y}$$

- The individual will allocate $\alpha$ percent of his income to good *x* and $\beta$ percent of his income to good *y*

# Cobb-Douglas Demand Functions

- The Cobb-Douglas utility function is limited in its ability to explain actual consumption behavior
  - the share of income devoted to particular goods often changes in response to changing economic conditions
- A more general functional form might be more useful in explaining consumption decisions

# CES Demand

- Assume that $\delta = 0.5$

$$U(x,y) = x^{0.5} + y^{0.5}$$

- Setting up the Lagrangian:

$$\mathbf{L} = x^{0.5} + y^{0.5} + \lambda(I - p_x x - p_y y)$$

- First-order conditions:

$$\partial \mathbf{L}/\partial x = 0.5x^{-0.5} - \lambda p_x = 0$$

$$\partial \mathbf{L}/\partial y = 0.5y^{-0.5} - \lambda p_y = 0$$

$$\partial \mathbf{L}/\partial \lambda = I - p_x x - p_y y = 0$$

# CES Demand

- This means that

$$(y/x)^{0.5} = p_x/p_y$$

- Substituting into the budget constraint, we can solve for the demand functions

$$x^* = \frac{I}{p_x[1+\frac{p_x}{p_y}]} \qquad y^* = \frac{I}{p_y[1+\frac{p_y}{p_x}]}$$

# CES Demand

- In these demand functions, the share of income spent on either *x* or *y* is not a constant
  - depends on the ratio of the two prices
- The higher is the relative price of *x* (or *y*), the smaller will be the share of income spent on *x* (or *y*)

# CES Demand

- If δ = -1,

$$U(x,y) = -x^{-1} - y^{-1}$$

- First-order conditions imply that

$$y/x = (p_x/p_y)^{0.5}$$

- The demand functions are

$$x^* = \frac{I}{p_x\left[1+\left(\frac{p_y}{p_x}\right)^{0.5}\right]} \qquad y^* = \frac{I}{p_y\left[1+\left(\frac{p_x}{p_y}\right)^{0.5}\right]}$$

# CES Demand

- If $\delta = -\infty$,

$$U(x,y) = Min(x,4y)$$

- The person will choose only combinations for which $x = 4y$
- This means that

$$I = p_x x + p_y y = p_x x + p_y (x/4)$$

$$I = (p_x + 0.25 p_y) x$$

# CES Demand

- Hence, the demand functions are

$$x^* = \frac{I}{p_x + 0.25p_y}$$

$$y^* = \frac{I}{4p_x + p_y}$$

# Indirect Utility Function

- It is often possible to manipulate first-order conditions to solve for optimal values of $x_1, x_2, \ldots, x_n$
- These optimal values will depend on the prices of all goods and income

$$x^*_1 = x_1(p_1, p_2, \ldots, p_n, I)$$
$$x^*_2 = x_2(p_1, p_2, \ldots, p_n, I)$$
$$\vdots$$
$$x^*_n = x_n(p_1, p_2, \ldots, p_n, I)$$

# Indirect Utility Function

- We can use the optimal values of the *x*s to find the indirect utility function

$$\text{maximum utility} = U(x^*_1, x^*_2, \ldots, x^*_n)$$

- Substituting for each $x^*_i$, we get

$$\text{maximum utility} = V(p_1, p_2, \ldots, p_n, I)$$

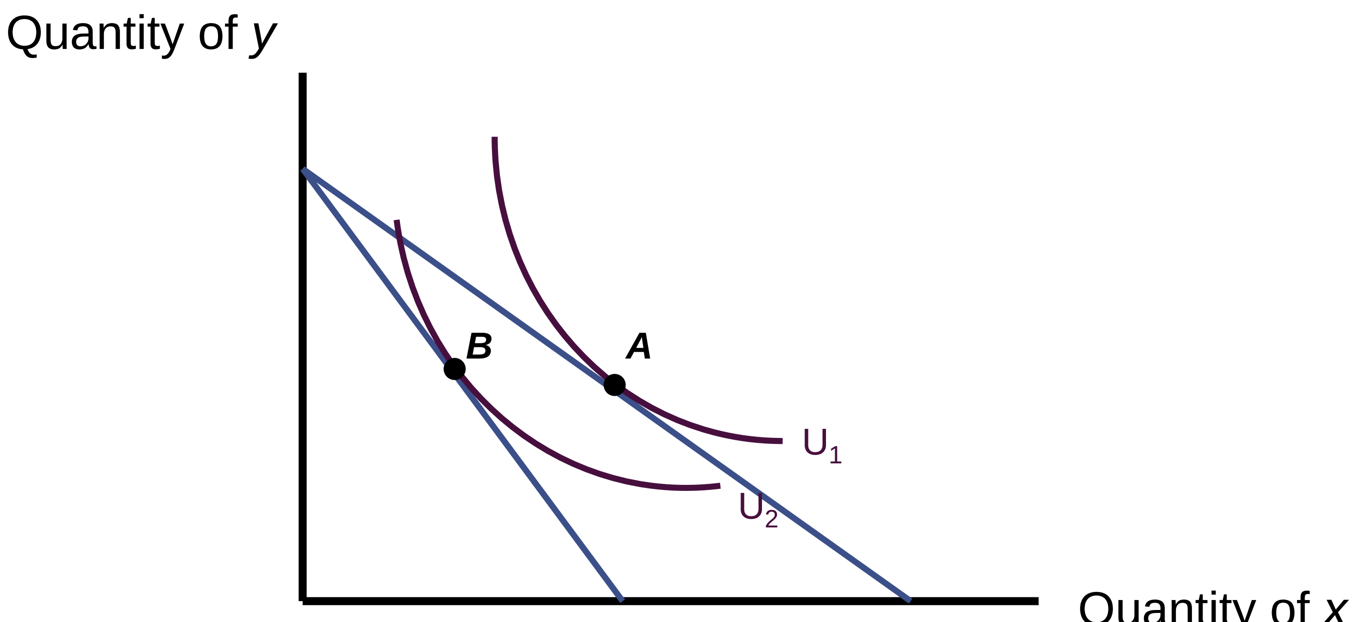
- The optimal level of utility will depend indirectly on prices and income
  - if either prices or income were to change, the maximum possible utility will change

# Lump Sum Principle（skip）

- Taxes on an individual’s general purchasing power are superior to taxes on a specific good
  - an income tax allows the individual to decide freely how to allocate remaining income
  - a tax on a specific good will reduce an individual’s purchasing power and distort his choices

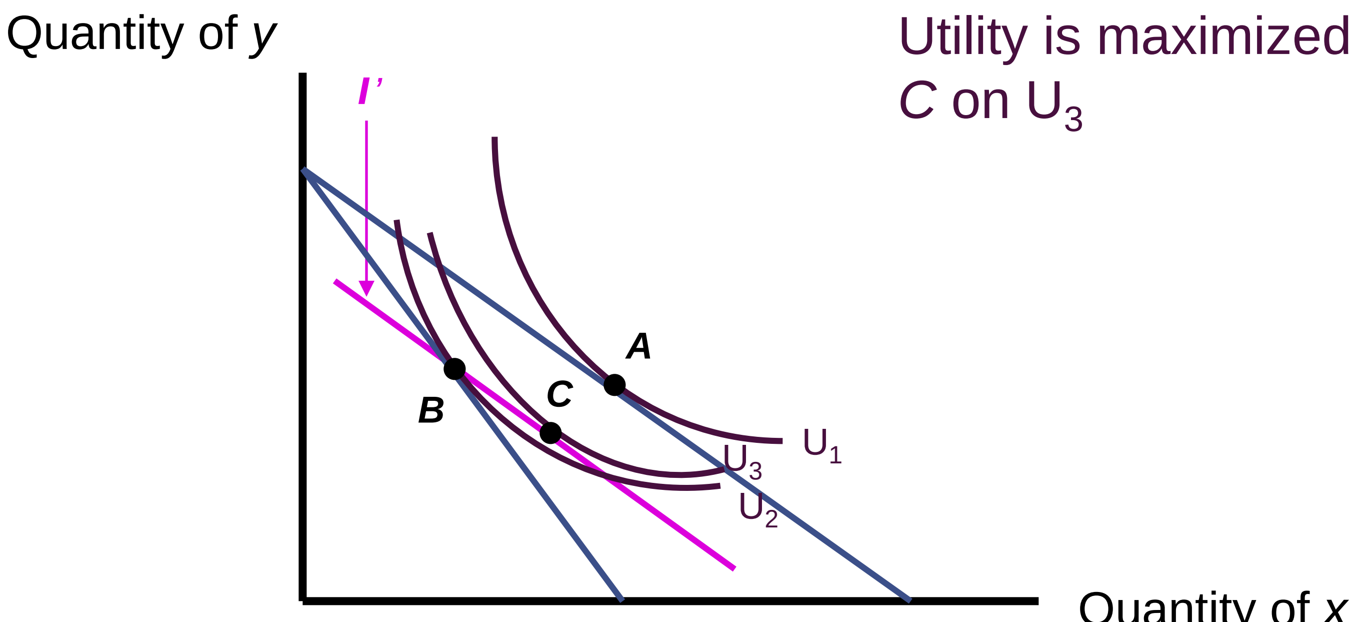
# The Lump Sum Principle

- A tax on good *x* would shift the utility-maximizing choice from point *A* to point *B*



# The Lump Sum Principle

- An income tax that collected the same amount would shift the budget constraint to *I*'




Utility is maximized now at point *C* on $U_3$

# Indirect Utility and the Lump Sum Principle

- If the utility function is Cobb-Douglas with $\alpha = \beta = 0.5$, we know that

$$x^* = \frac{I}{2p_x} \qquad y^* = \frac{I}{2p_y}$$

- So the indirect utility function is

$$V(p_x, p_y, I) = (x^*)^{0.5}(y^*)^{0.5} = \frac{I}{2p_x^{0.5}p_y^{0.5}}$$

# Indirect Utility and the Lump Sum Principle

- If a tax of \$1 was imposed on good $x$
  - the individual will purchase $x^*=2$
  - indirect utility will fall from 2 to 1.41
- An equal-revenue tax will reduce income to \$6
  - indirect utility will fall from 2 to 1.5

# Indirect Utility and the Lump Sum Principle

- If the utility function is fixed proportions with $U = Min(x,4y)$, we know that

$$x^* = \frac{I}{p_x + 0.25p_y} \qquad y^* = \frac{I}{4p_x + p_y}$$

- So the indirect utility function is

$$V(p_x, p_y, I) = Min(x^*, 4y^*) = x^* = \frac{I}{p_x + 0.25p_y}$$

$$= 4y^* = \frac{4}{4p_x + p_y} = \frac{I}{p_x + 0.25p_y}$$

# Indirect Utility and the Lump Sum Principle

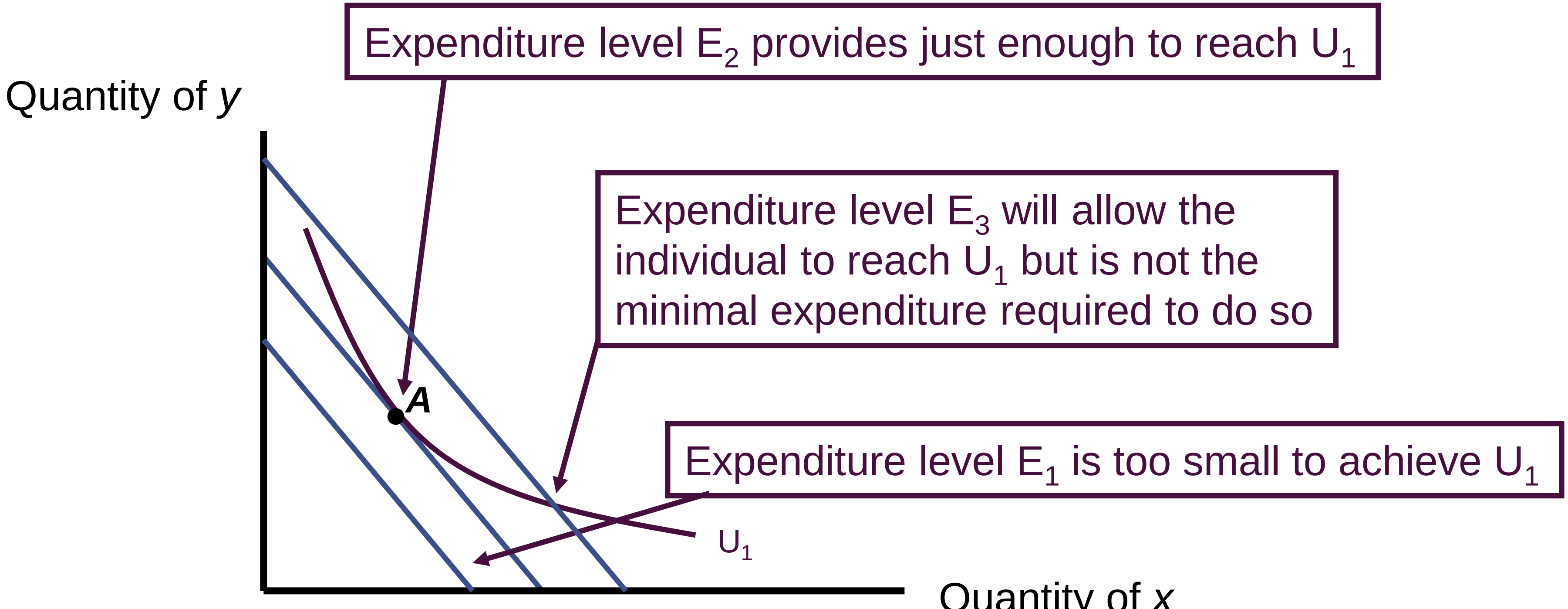
- If a tax of $1 was imposed on good *x*
  - indirect utility will fall from 4 to 8/3
- An equal-revenue tax will reduce income to $16/3
  - indirect utility will fall from 4 to 8/3
- Since preferences are rigid, the tax on *x* does not distort choices

# Expenditure Minimization

- Dual minimization problem for utility maximization
  - allocating income in such a way as to achieve a given level of utility with the minimal expenditure
  - this means that the goal and the constraint have been reversed

# Expenditure Minimization

- Point *A* is the solution to the dual problem



# Expenditure Minimization

- The individual's problem is to choose $x_1, x_2, \ldots, x_n$ to minimize

$$\text{total expenditures} = E = p_1x_1 + p_2x_2 + \ldots + p_nx_n$$

subject to the constraint

$$\text{utility} = U_1 = U(x_1, x_2, \ldots, x_n)$$

- The optimal amounts of $x_1, x_2, \ldots, x_n$ will depend on the prices of the goods and the required utility level

# Expenditure Function （支出函数）

- The <u>expenditure function</u> shows the minimal expenditures necessary to achieve a given utility level for a particular set of prices

$$\text{minimal expenditures} = E(p_1,p_2,\ldots,p_n,U)$$

- The expenditure function and the indirect utility function are inversely related
  - both depend on market prices but involve different constraints

# Two Expenditure Functions

- The indirect utility function in the two-good, Cobb-Douglas case is

$$V(p_x, p_y, I) = \frac{I}{2p_x^{0.5} p_y^{0.5}}$$

- If we interchange the role of utility and income (expenditure), we will have the expenditure function

$$E(p_x, p_y, U) = 2p_x^{0.5} p_y^{0.5} U$$

# Two Expenditure Functions

- For the fixed-proportions case （ $\delta = -\infty$ ）, the indirect utility function is

$$V(p_x, p_y, I) = \frac{I}{p_x + 0.25 p_y}$$

- If we again switch the role of utility and expenditures, we will have the expenditure function

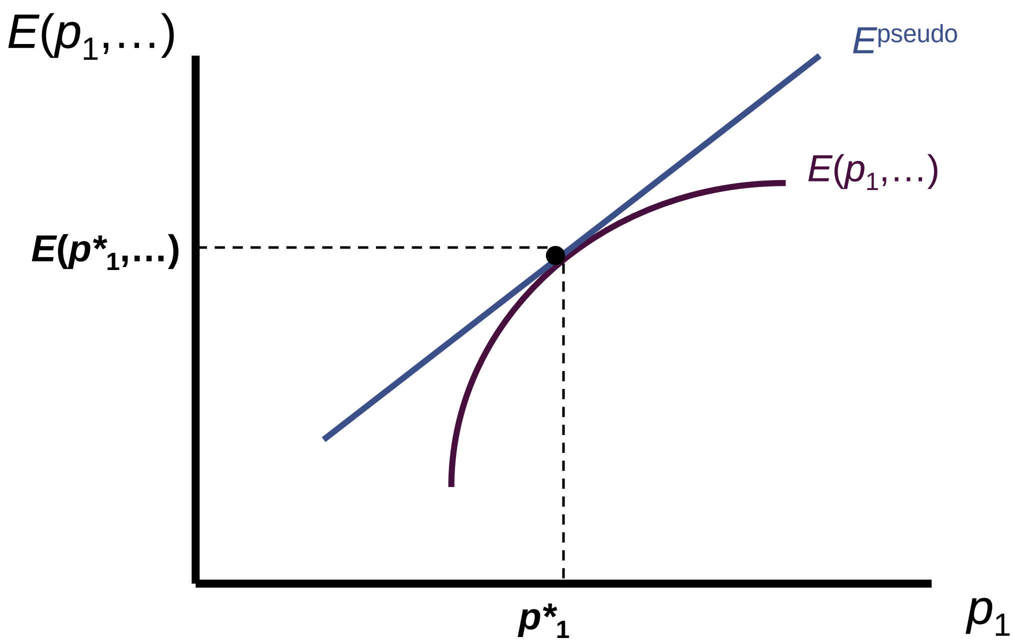
$$E(p_x, p_y, U) = (p_x + 0.25 p_y) U$$

# Properties of Expenditure Functions

- Homogeneity
  - a doubling of all prices will precisely double the value of required expenditures
    - homogeneous of degree one
- Nondecreasing in prices
  - $\partial E/\partial p_i \geq 0$ for every good, $i$
- Concave in prices

# Concavity of Expenditure Function

At $p^*_1$, the person spends $E(p^*_1,\ldots)$




If he continues to buy the same set of goods as $p^*_1$ changes, his expenditure function would be $E^{pseudo}$

Since his consumption pattern will likely change, actual expenditures will be less than $E^{pseudo}$ such as $E(p_1,\ldots)$

# Important Points to Note:

- To reach a constrained maximum, an individual should:
  - spend all available income
  - choose a commodity bundle such that the *MRS* between any two goods is equal to the ratio of the goods' prices
    - the individual will equate the ratios of the marginal utility to price for every good that is actually consumed

# Important Points to Note:

- Tangency conditions are only first-order conditions
  - the individual's indifference map must exhibit diminishing *MRS*
  - the utility function must be strictly quasi-concave

# Important Points to Note:

- Tangency conditions must also be modified to allow for corner solutions
  - the ratio of marginal utility to price will be below the common marginal benefit-marginal cost ratio for goods actually bought

# Important Points to Note:

- The individual's optimal choices implicitly depend on the parameters of his budget constraint
  - choices observed will be implicit functions of prices and income
  - utility will also be an indirect function of prices and income

# Important Points to Note:

- The dual problem to the constrained utility-maximization problem is to minimize the expenditure required to reach a given utility target
  - yields the same optimal solution as the primary problem
  - leads to expenditure functions in which spending is a function of the utility target and prices