Chapter 7

PRODUCTION FUNCTIONS

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Production Function

- The firm's <u>production function</u> for a particular good (*q*) shows the maximum amount of the good that can be produced using alternative combinations of capital (*k*) and labor (*l*)

$$q = f(k,l)$$

Marginal Physical Product

- To study variation in a single input, we define <u>marginal physical product</u> as the additional output that can be produced by employing one more unit of that input while holding other inputs constant

$$\text{marginal physical product of capital} = MP_k = \frac{\partial q}{\partial k} = f_k$$

$$\text{marginal physical product of labor} = MP_l = \frac{\partial q}{\partial l} = f_l$$

Diminishing Marginal Productivity (边际生产力递减)

- The marginal physical product of an input depends on how much of that input is used
- In general, we assume diminishing marginal productivity

$$\frac{\partial MP_k}{\partial k} = \frac{\partial^2 f}{\partial k^2} = f_{kk} = f_{11} < 0 \qquad \frac{\partial MP_l}{\partial l} = \frac{\partial^2 f}{\partial l^2} = f_{ll} = f_{22} < 0$$

Diminishing Marginal Productivity

- Because of diminishing marginal productivity, 19th century economist Thomas Malthus worried about the effect of population growth on labor productivity
- But changes in the marginal productivity of labor over time also depend on changes in other inputs such as capital
  - we need to consider f_{lk} which is often > 0

Average Physical Product

- Labor productivity is often measured by average productivity

$$AP_l = \frac{\text{output}}{\text{labor input}} = \frac{q}{l} = \frac{f(k,l)}{l}$$

- Note that AP_l also depends on the amount of capital employed

A Two-Input Production Function

- Suppose the production function for flyswatters can be represented by

$$q = f(k,l) = 600k^2l^2 - k^3l^3$$

- To construct MP_l and AP_l, we must assume a value for k
  - let $k = 10$
- The production function becomes

$$q = 60{,}000l^2 - 1000l^3$$

A Two-Input Production Function

- The marginal productivity function is

$$MP_l = \partial q/\partial l = 120{,}000l - 3000l^2$$

which diminishes as l increases

- This implies that q has a maximum value:

$$120{,}000l - 3000l^2 = 0$$

$$40l = l^2$$

$$l = 40$$

- Labor input beyond $l = 40$ reduces output

A Two-Input Production Function

- To find average productivity, we hold k=10 and solve

$$AP_l = q/l = 60{,}000l - 1000l^2$$

- AP_l reaches its maximum where

$$\partial AP_l/\partial l = 60{,}000 - 2000l = 0$$

$$l = 30$$

A Two-Input Production Function

- In fact, when $l = 30$, both AP_l and MP_l are equal to 900,000

- Thus, when AP_l is at its maximum, AP_l and MP_l are equal

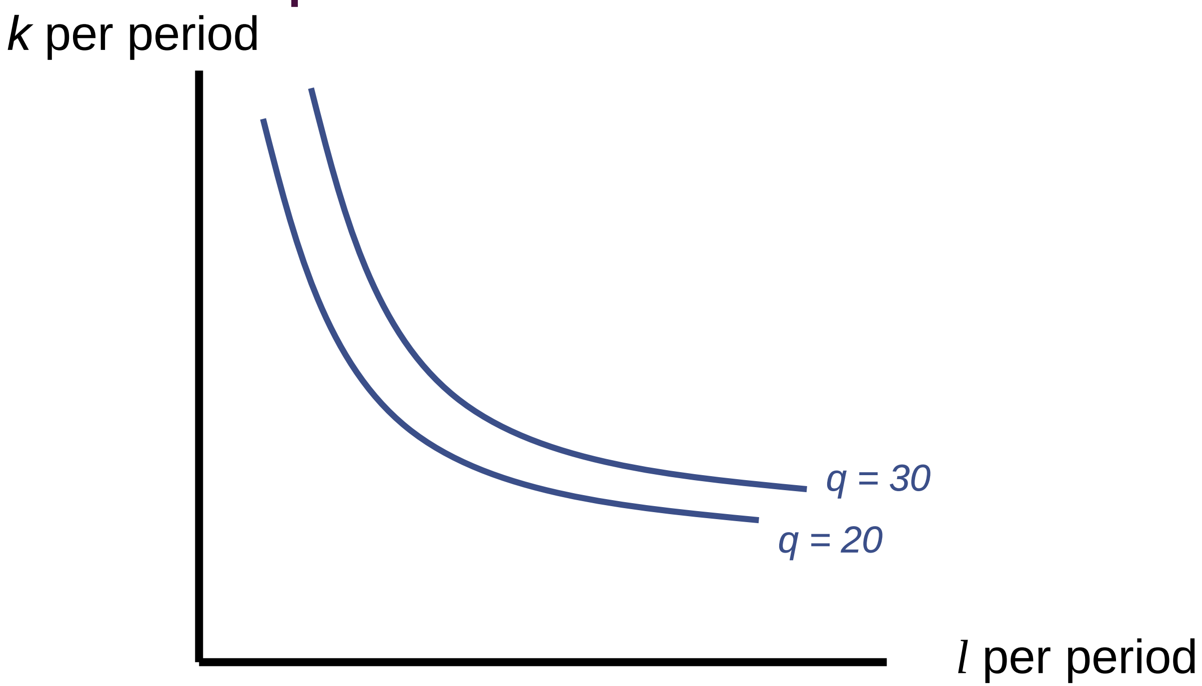
Isoquant Maps

- To illustrate the possible substitution of one input for another, we use an isoquant map（等产量线图）
- An isoquant shows those combinations of k and l that can produce a given level of output (q_0)

$$f(k,l) = q_0$$

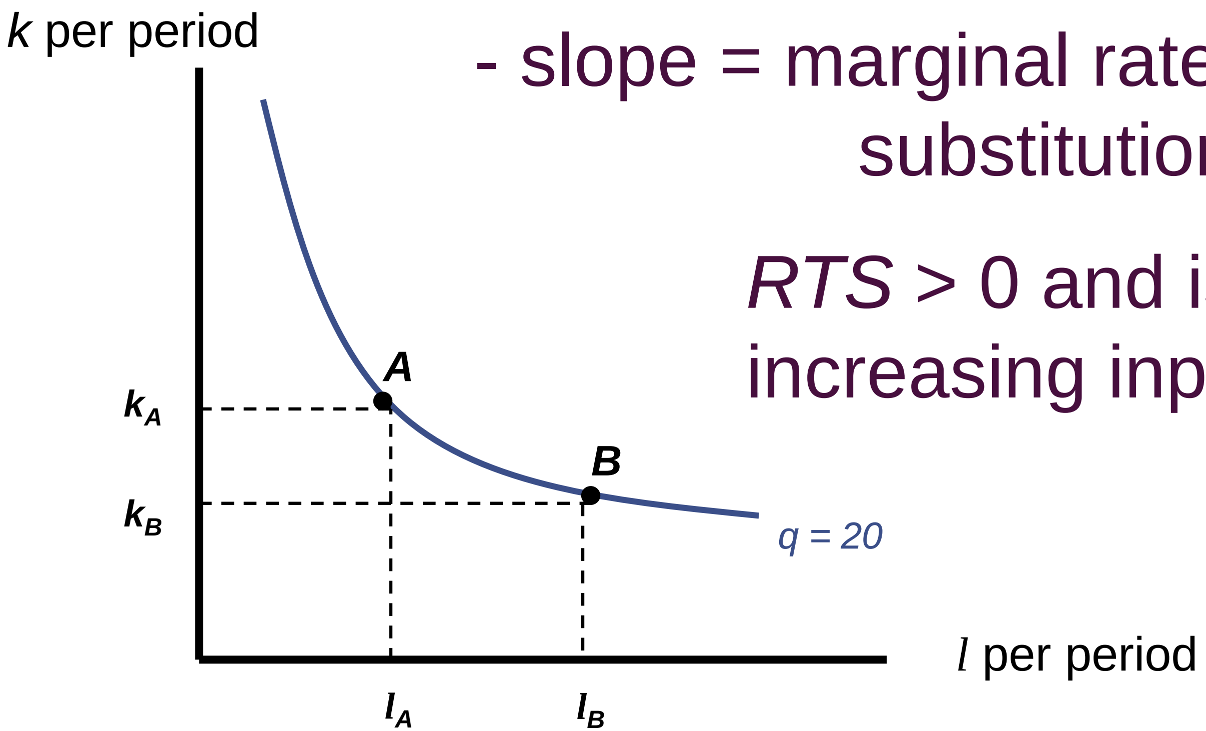
Isoquant Map

- Each isoquant represents a different level of output
  - output rises as we move northeast



Marginal Rate of Technical Substitution (*RTS*)

- The slope of an isoquant shows the rate at which *l* can be substituted for *k*



- slope = marginal rate of technical substitution (*RTS*)

RTS > 0 and is diminishing for increasing inputs of labor

Marginal Rate of Technical Substitution (*RTS*) 边际技术替代率

- The marginal rate of technical substitution (*RTS*) shows the rate at which labor can be substituted for capital while holding output constant along an isoquant

$$RTS\,(l \text{ for } k) = \left.\frac{-dk}{dl}\right|_{q=q_0}$$

RTS and Marginal Productivities

- Take the total differential of the production function:

$$dq = \frac{\partial f}{\partial l} \cdot dl + \frac{\partial f}{\partial k} \cdot dk = MP_l \cdot dl + MP_k \cdot dk$$

- Along an isoquant *dq* = 0, so

$$MP_l \cdot dl = -MP_k \cdot dk$$

$$RTS\,(l \text{ for } k) = \left.\frac{-dk}{dl}\right|_{q=q_0} = \frac{MP_l}{MP_k}$$

RTS and Marginal Productivities

- Because MP_l and MP_k will both be nonnegative, *RTS* will be positive (or zero)
- However, it is generally not possible to derive a diminishing *RTS* from the assumption of diminishing marginal productivity alone

RTS and Marginal Productivities

- To show that isoquants are convex, we would like to show that $d(RTS)/dl < 0$
- Since $RTS = f_l/f_k$

$$\frac{dRTS}{dl} = \frac{d(f_l / f_k)}{dl}$$

$$\frac{dRTS}{dl} = \frac{[f_k(f_{ll} + f_{lk} \cdot dk/dl) - f_l(f_{kl} + f_{kk} \cdot dk/dl)]}{(f_k)^2}$$

RTS and Marginal Productivities

- Using the fact that $dk/dl = -f_l/f_k$ along an isoquant and Young's theorem ($f_{kl} = f_{lk}$)

$$\frac{dRTS}{dl} = \frac{(f_k^2 f_{ll} - 2f_k f_l f_{kl} + f_l^2 f_{kk})}{(f_k)^3}$$

- Because we have assumed $f_k > 0$, the denominator is positive
- Because f_{ll} and f_{kk} are both assumed to be negative, the ratio will be negative if f_{kl} is positive

RTS and Marginal Productivities

- Intuitively, it seems reasonable that $f_{kl} = f_{lk}$ should be positive
  - if workers have more capital, they will be more productive
- But some production functions have $f_{kl} < 0$ over some input ranges
  - when we assume diminishing *RTS* we are assuming that MP_l and MP_k diminish quickly enough to compensate for any possible negative cross-productivity effects

A Diminishing *RTS*

- Suppose the production function is

$$q = f(k,l) = 600k^2l^2 - k^3l^3$$

- For this production function

$$MP_l = f_l = 1200k^2l - 3k^3l^2$$

$$MP_k = f_k = 1200kl^2 - 3k^2l^3$$

  – these marginal productivities will be positive for values of *k* and *l* for which $kl < 400$

A Diminishing *RTS*

- Because

$$f_{ll} = 1200k^2 - 6k^3l$$

$$f_{kk} = 1200l^2 - 6kl^3$$

this production function exhibits diminishing marginal productivities for sufficiently large values of *k* and *l*

– f_{ll} and $f_{kk} < 0$ if $kl > 200$

A Diminishing *RTS*

- Cross differentiation of either of the marginal productivity functions yields

$$f_{kl} = f_{lk} = 2400kl - 9k^2l^2$$

which is positive only for $kl < 266$

A Diminishing *RTS*

- Thus, for this production function, *RTS* is diminishing throughout the range of k and l where marginal productivities are positive
  - for higher values of k and l, the diminishing marginal productivities are sufficient to overcome the influence of a negative value for f_{kl} to ensure convexity of the isoquants

Returns to Scale 规模回报

- How does output respond to increases in all inputs together?
  - suppose that all inputs are doubled, would output double?
- Returns to scale have been of interest to economists since the days of Adam Smith

Returns to Scale

- Smith identified two forces that come into operation as inputs are doubled
  – greater division of labor and specialization of function
  – loss in efficiency because management may become more difficult given the larger scale of the firm

Returns to Scale

- If the production function is given by $q = f(k,l)$ and all inputs are multiplied by the same positive constant ($t > 1$), then

Effect on Output	Returns to Scale
$f(tk,tl) = tf(k,l)$	Constant
$f(tk,tl) < tf(k,l)$	Decreasing
$f(tk,tl) > tf(k,l)$	Increasing

Returns to Scale

- It is possible for a production function to exhibit constant returns to scale for some levels of input usage and increasing or decreasing returns for other levels
  - economists refer to the degree of returns to scale with the implicit notion that only a fairly narrow range of variation in input usage and the related level of output is being considered

Constant Returns to Scale

- Constant returns-to-scale production functions are homogeneous of degree one in inputs

$$f(tk,tl) = t^1 f(k,l) = tq$$

- This implies that the marginal productivity functions are homogeneous of degree zero
  - if a function is homogeneous of degree *k*, its derivatives are homogeneous of degree *k*-1

Constant Returns to Scale

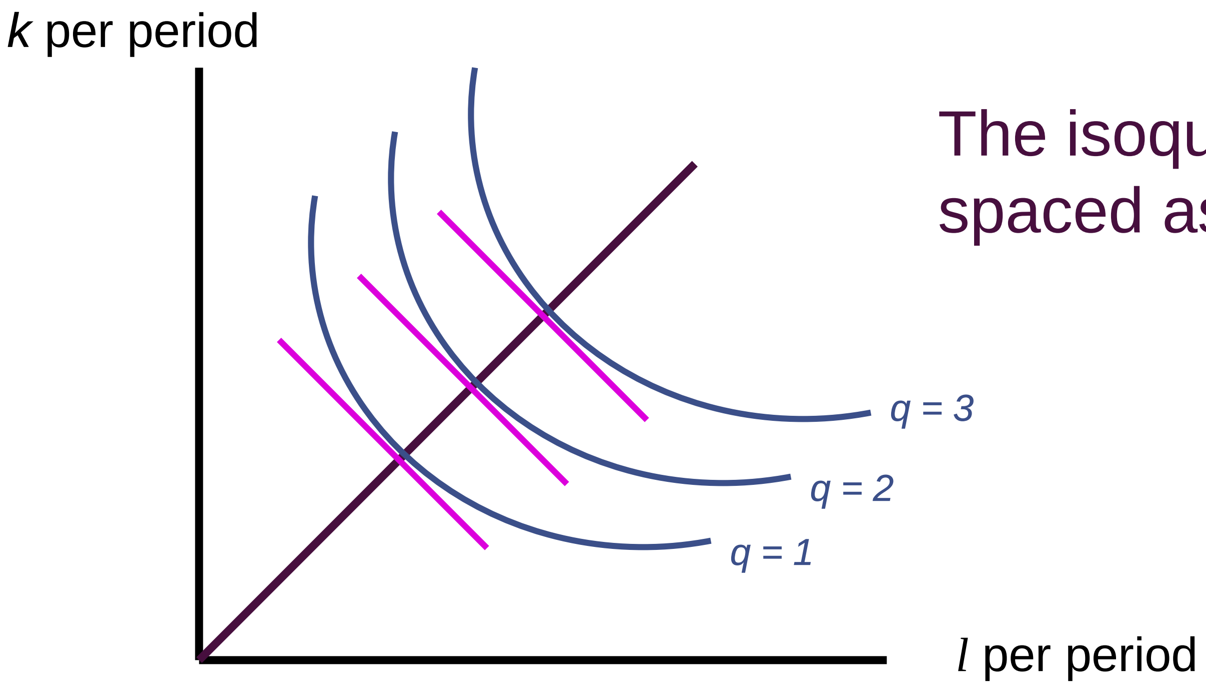
- The marginal productivity of any input depends on the ratio of capital and labor (not on the absolute levels of these inputs)
- The *RTS* between *k* and *l* depends only on the ratio of *k* to *l*, not the scale of operation

Constant Returns to Scale

- The production function will be homothetic
- Geometrically, all of the isoquants are radial expansions of one another

Constant Returns to Scale

- Along a ray from the origin (constant *k/l*), the *RTS* will be the same on all isoquants



The isoquants are equally spaced as output expands

Returns to Scale

- Returns to scale can be generalized to a production function with *n* inputs

$$q = f(x_1, x_2, \ldots, x_n)$$

- If all inputs are multiplied by a positive constant *t*, we have

$$f(tx_1, tx_2, \ldots, tx_n) = t^k f(x_1, x_2, \ldots, x_n) = t^k q$$

  - If $k = 1$, we have constant returns to scale
  - If $k < 1$, we have decreasing returns to scale
  - If $k > 1$, we have increasing returns to scale

Elasticity of Substitution

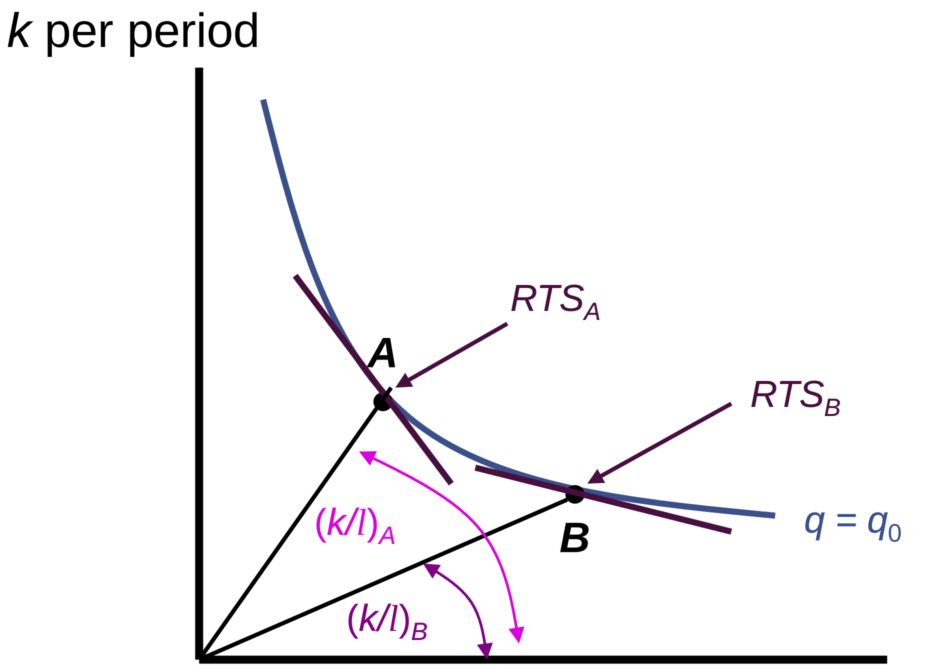
- The elasticity of substitution (σ) measures the proportionate change in *k*/*l* relative to the proportionate change in the *RTS* along an isoquant

$$\sigma = \frac{\%\Delta(k/l)}{\%\Delta RTS} = \frac{d(k/l)}{dRTS} \cdot \frac{RTS}{k/l} = \frac{\partial \ln(k/l)}{\partial \ln RTS}$$

- The value of σ will always be positive because *k*/*l* and *RTS* move in the same direction

Elasticity of Substitution

- Both *RTS* and *k*/*l* will change as we move from point *A* to point *B*



σ is the ratio of these proportional changes

σ measures the curvature of the isoquant

Elasticity of Substitution

- If σ is high, the *RTS* will not change much relative to *k*/*l*
  - the isoquant will be relatively flat
- If σ is low, the *RTS* will change by a substantial amount as *k*/*l* changes
  - the isoquant will be sharply curved
- It is possible for σ to change along an isoquant or as the scale of production changes

Elasticity of Substitution

- Generalizing the elasticity of substitution to the many-input case raises several complications
  - if we define the elasticity of substitution between two inputs to be the proportionate change in the ratio of the two inputs to the proportionate change in *RTS*, we need to hold output <u>and</u> the levels of other inputs constant

The Linear Production Function

- Suppose that the production function is

$$q = f(k,l) = ak + bl$$

- This production function exhibits constant returns to scale

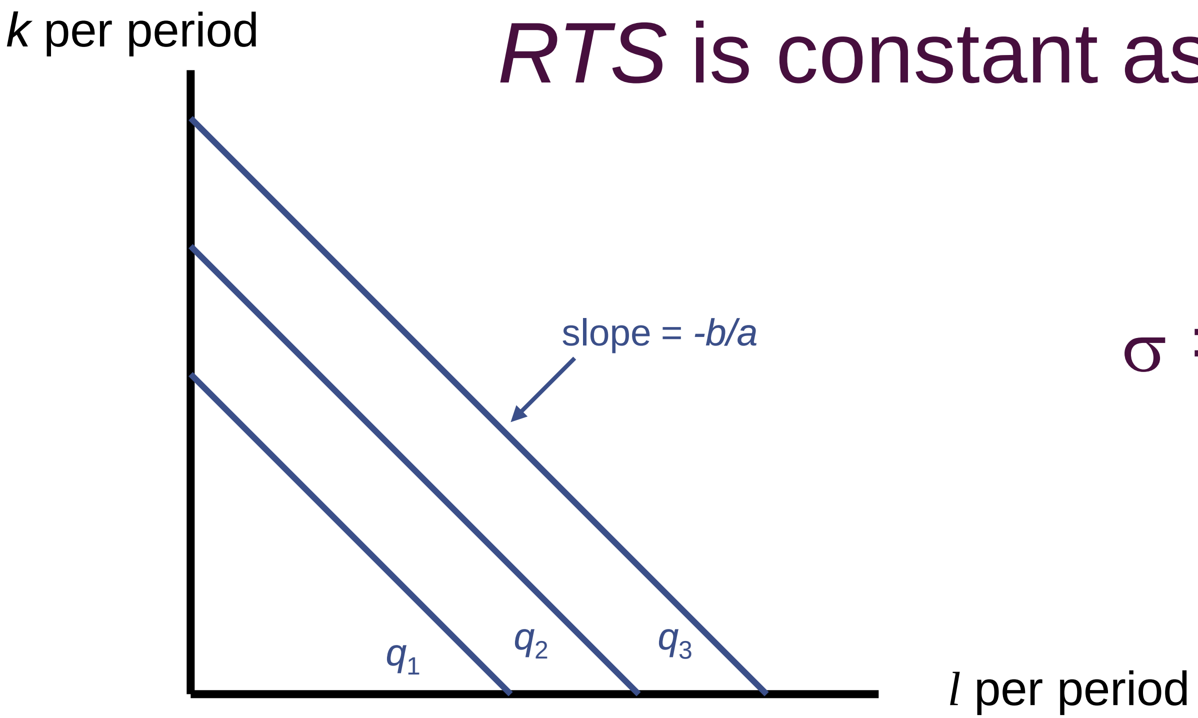
$$f(tk,tl) = atk + btl = t(ak + bl) = tf(k,l)$$

- All isoquants are straight lines
  - *RTS* is constant
  - $\sigma = \infty$

The Linear Production Function

Capital and labor are perfect substitutes

RTS is constant as *k*/*l* changes



$$\sigma = \infty$$

Fixed Proportions

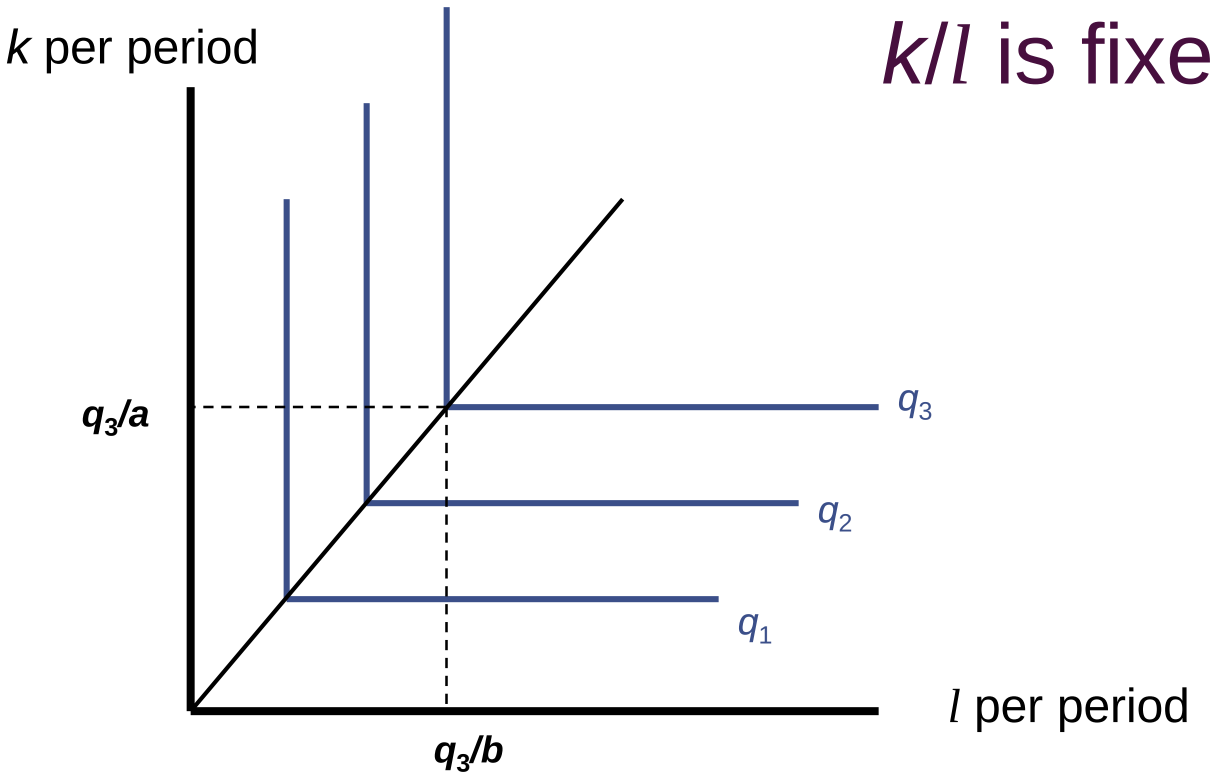
- Suppose that the production function is

$$q = \min(ak, bl) \quad a,b > 0$$

- Capital and labor must always be used in a fixed ratio
  - the firm will always operate along a ray where k/l is constant
- Because k/l is constant, $\sigma = 0$

Fixed Proportions

No substitution between labor and capital is possible



k/l is fixed at b/a

$\sigma = 0$

Cobb-Douglas Production Function

- Suppose that the production function is

$$q = f(k,l) = Ak^a l^b \quad A,a,b > 0$$

- This production function can exhibit any returns to scale

$$f(tk,tl) = A(tk)^a(tl)^b = At^{a+b} k^a l^b = t^{a+b}f(k,l)$$

  – if $a + b = 1 \Rightarrow$ constant returns to scale
  – if $a + b > 1 \Rightarrow$ increasing returns to scale
  – if $a + b < 1 \Rightarrow$ decreasing returns to scale

Cobb-Douglas Production Function

- The Cobb-Douglas production function is linear in logarithms

$$\ln q = \ln A + a \ln k + b \ln l$$

  – a is the elasticity of output with respect to k
  – b is the elasticity of output with respect to l

CES Production Function

- Suppose that the production function is

$$q = f(k,l) = [k^{\rho} + l^{\rho}]^{\gamma/\rho} \quad \rho \leq 1,\ \rho \neq 0,\ \gamma > 0$$

  - $\gamma > 1 \Rightarrow$ increasing returns to scale
  - $\gamma < 1 \Rightarrow$ decreasing returns to scale
- For this production function

$$\sigma = 1/(1-\rho)$$

  - $\rho = 1 \Rightarrow$ linear production function
  - $\rho = -\infty \Rightarrow$ fixed proportions production function
  - $\rho = 0 \Rightarrow$ Cobb-Douglas production function

A Generalized Leontief Production Function

- Suppose that the production function is

$$q = f(k,l) = k + l + 2(kl)^{0.5}$$

- Marginal productivities are

$$f_k = 1 + (k/l)^{-0.5}$$

$$f_l = 1 + (k/l)^{0.5}$$

- Thus,

$$RTS = \frac{f_l}{f_k} = \frac{1 + (k/l)^{0.5}}{1 + (k/l)^{-0.5}}$$

Technical Progress

- Methods of production change over time
- Following the development of superior production techniques, the same level of output can be produced with fewer inputs
  - the isoquant shifts in

Technical Progress

- Suppose that the production function is

$$q = A(t)f(k,l)$$

where $A(t)$ represents all influences that go into determining q other than k and l

  – changes in A over time represent technical progress

    - A is shown as a function of time (t)
    - $dA/dt > 0$

Technical Progress

- Differentiating the production function with respect to time we get

$$\frac{dq}{dt} = \frac{dA}{dt} \cdot f(k,l) + A \cdot \frac{df(k,l)}{dt}$$

$$\frac{dq}{dt} = \frac{dA}{dt} \cdot \frac{q}{A} + \frac{q}{f(k,l)} \left[\frac{\partial f}{\partial k} \cdot \frac{dk}{dt} + \frac{\partial f}{\partial l} \cdot \frac{dl}{dt} \right]$$

Technical Progress

- Dividing by *q* gives us

$$\frac{dq/dt}{q} = \frac{dA/dt}{A} + \frac{\partial f/\partial k}{f(k,l)} \cdot \frac{dk}{dt} + \frac{\partial f/\partial l}{f(k,l)} \cdot \frac{dl}{dt}$$

$$\frac{dq/dt}{q} = \frac{dA/dt}{A} + \frac{\partial f}{\partial k} \cdot \frac{k}{f(k,l)} \cdot \frac{dk/dt}{k} + \frac{\partial f}{\partial l} \cdot \frac{l}{f(k,l)} \cdot \frac{dl/dt}{l}$$

Technical Progress

- For any variable x, $[(dx/dt)/x]$ is the proportional growth rate in x
  - denote this by G_x
- Then, we can write the equation in terms of growth rates

$$G_q = G_A + \frac{\partial f}{\partial k} \cdot \frac{k}{f(k,l)} \cdot G_k + \frac{\partial f}{\partial l} \cdot \frac{l}{f(k,l)} \cdot G_l$$

Technical Progress

- Since

$$\frac{\partial f}{\partial k} \cdot \frac{k}{f(k,l)} = \frac{\partial q}{\partial k} \cdot \frac{k}{q} = e_{q,k}$$

$$\frac{\partial f}{\partial l} \cdot \frac{l}{f(k,l)} = \frac{\partial q}{\partial l} \cdot \frac{l}{q} = e_{q,l}$$

$$G_q = G_A + e_{q,k}G_k + e_{q,l}G_l$$

Technical Progress in the Cobb-Douglas Function

- Suppose that the production function is

$$q = A(t)f(k,l) = A(t)k^{\alpha}l^{1-\alpha}$$

- If we assume that technical progress occurs at a constant exponential (θ) then

$$A(t) = Ae^{\theta\text{-}t}$$

$$q = Ae^{\theta\text{-}t}k^{\alpha}l^{1-\alpha}$$

Technical Progress in the Cobb-Douglas Function

- Taking logarithms and differentiating with respect to *t* gives the growth equation

$$\frac{\partial \ln q}{\partial t} = \frac{\partial \ln q}{\partial q} \cdot \frac{\partial q}{\partial t} = \frac{\partial q / \partial t}{q} = G_q$$

Technical Progress in the Cobb-Douglas Function

$$G_q = \frac{\partial(\ln A + \theta t + \alpha \ln k + (1-\alpha)\ln l)}{\partial t}$$

$$= \theta + \alpha \cdot \frac{\partial \ln k}{\partial t} + (1-\alpha) \cdot \frac{\partial \ln l}{\partial t} = \theta + \alpha G_k + (1-\alpha) G_l$$

Important Points to Note:

- If all but one of the inputs are held constant, a relationship between the single variable input and output can be derived
  - the marginal physical productivity is the change in output resulting from a one-unit increase in the use of the input
    - assumed to decline as use of the input increases

Important Points to Note:

- The entire production function can be illustrated by an isoquant map
  - the slope of an isoquant is the marginal rate of technical substitution (*RTS*)
    - it shows how one input can be substituted for another while holding output constant
    - it is the ratio of the marginal physical productivities of the two inputs

Important Points to Note:

- Isoquants are usually assumed to be convex
  - they obey the assumption of a diminishing *RTS*
    - this assumption cannot be derived exclusively from the assumption of diminishing marginal productivity
    - one must be concerned with the effect of changes in one input on the marginal productivity of other inputs

Important Points to Note:

- The returns to scale exhibited by a production function record how output responds to proportionate increases in all inputs
  - if output increases proportionately with input use, there are constant returns to scale

Important Points to Note:

- The elasticity of substitution (σ) provides a measure of how easy it is to substitute one input for another in production
  - a high σ implies nearly straight isoquants
  - a low σ implies that isoquants are nearly L-shaped

Important Points to Note:

- Technical progress shifts the entire production function and isoquant map
  - technical improvements may arise from the use of more productive inputs or better methods of economic organization