# Chapter 8

## COST FUNCTIONS

# Definitions of Costs

- It is important to differentiate between accounting cost and economic cost
  - the accountant’s view of cost stresses out-of-pocket expenses, historical costs, depreciation, and other bookkeeping entries
  - economists focus more on opportunity cost 机会成本

# Definitions of Costs

- Labor Costs
  - to accountants, expenditures on labor are current expenses and hence costs of production
  - to economists, labor is an explicit cost
    - labor services are contracted at some hourly wage ($w$) and it is assumed that this is also what the labor could earn in alternative employment

# Definitions of Costs

- Capital Costs
  - accountants use the historical price of the capital and apply some depreciation rule to determine current costs
  - economists refer to the capital's original price as a "sunk cost" and instead regard the implicit cost of the capital to be what someone else would be willing to pay for its use
    - we will use *v* to denote the rental rate for capital

# Definitions of Costs

- Costs of Entrepreneurial Services
  - accountants believe that the owner of a firm is entitled to all profits
    - revenues or losses left over after paying all input costs
  - economists consider the opportunity costs of time and funds that owners devote to the operation of their firms
    - part of accounting profits would be considered as entrepreneurial costs by economists

# Economic Cost

- The economic cost of any input is the payment required to keep that input in its present employment
  - the remuneration the input would receive in its best alternative employment

# Two Simplifying Assumptions

- There are only two inputs
  - homogeneous labor (*l*), measured in labor-hours
  - homogeneous capital (*k*), measured in machine-hours
    - entrepreneurial costs are included in capital costs
- Inputs are hired in perfectly competitive markets
  - firms are price takers in input markets

# Economic Profits

- Total costs for the firm are given by

$$\text{total costs} = C = wl + vk$$

- Total revenue for the firm is given by

$$\text{total revenue} = pq = pf(k,l)$$

- Economic profits ($\pi$) are equal to

$$\pi = \text{total revenue} - \text{total cost}$$

$$\pi = pq - wl - vk$$

$$\pi = pf(k,l) - wl - vk$$

# Economic Profits

- Economic profits are a function of the amount of capital and labor employed
  - we could examine how a firm would choose $k$ and $l$ to maximize profit
    - "derived demand" theory of labor and capital inputs
  - for now, we will assume that the firm has already chosen its output level ($q_0$) and wants to minimize its costs

# Cost-Minimizing Input Choices

- To minimize the cost of producing a given level of output, a firm should choose a point on the isoquant at which the *RTS* is equal to the ratio *w*/*v*
  - it should equate the rate at which *k* can be traded for *l* in the productive process to the rate at which they can be traded in the marketplace

# Cost-Minimizing Input Choices

- Mathematically, we seek to minimize total costs given $q = f(k,l) = q_0$
- Setting up the Lagrangian:

$$\mathbf{L} = wl + vk + \lambda[q_0 - f(k,l)]$$

- First order conditions are

$$\partial\mathbf{L}/\partial l = w - \lambda(\partial f/\partial l) = 0$$

$$\partial\mathbf{L}/\partial k = v - \lambda(\partial f/\partial k) = 0$$

$$\partial\mathbf{L}/\partial\lambda = q_0 - f(k,l) = 0$$

# Cost-Minimizing Input Choices

- Dividing the first two conditions we get

$$\frac{w}{v} = \frac{\partial f / \partial l}{\partial f / \partial k} = RTS\ (l \text{ for } k)$$

- The cost-minimizing firm should equate the *RTS* for the two inputs to the ratio of their prices

# Cost-Minimizing Input Choices

- Cross-multiplying, we get

$$\frac{f_k}{v} = \frac{f_l}{w}$$

- For costs to be minimized, the marginal productivity per dollar spent should be the same for all inputs

# Cost-Minimizing Input Choices

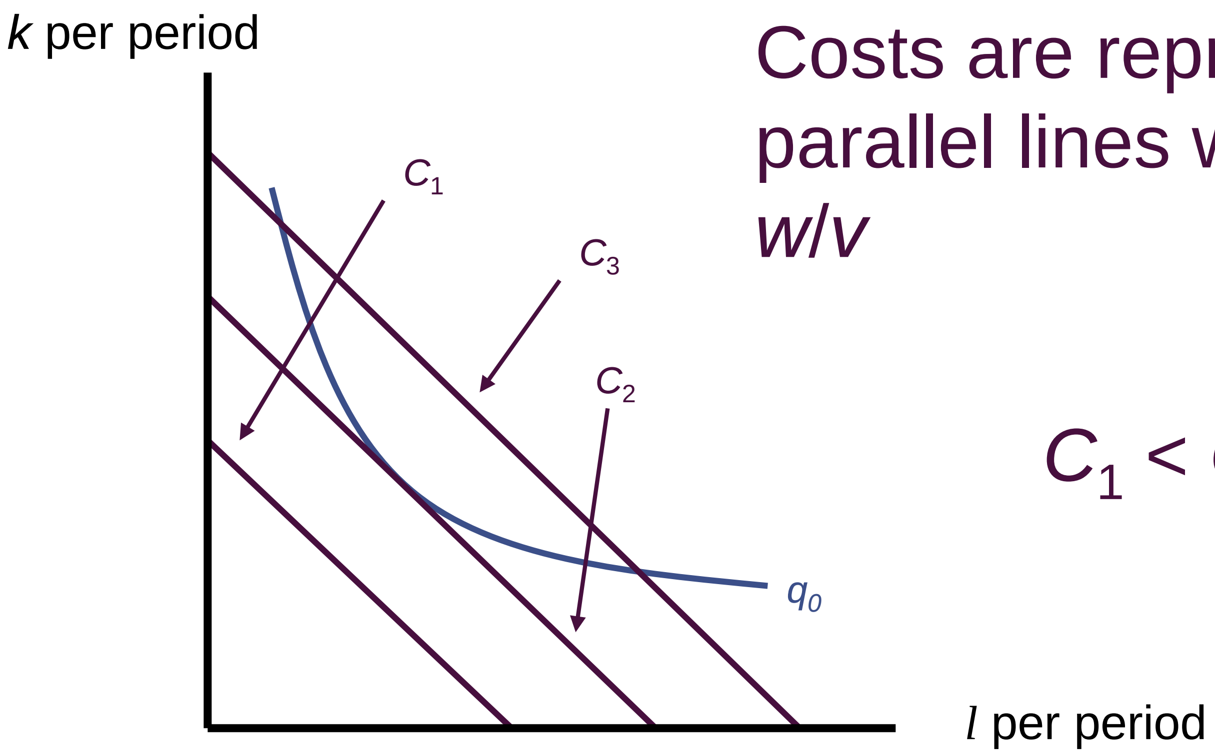
- Note that this equation's inverse is also of interest

$$\frac{w}{f_l} = \frac{v}{f_k} = \lambda$$

- The Lagrangian multiplier shows how much in extra costs would be incurred by increasing the output constraint slightly

# Cost-Minimizing Input Choices

Given output $q_0$, we wish to find the least costly point on the isoquant

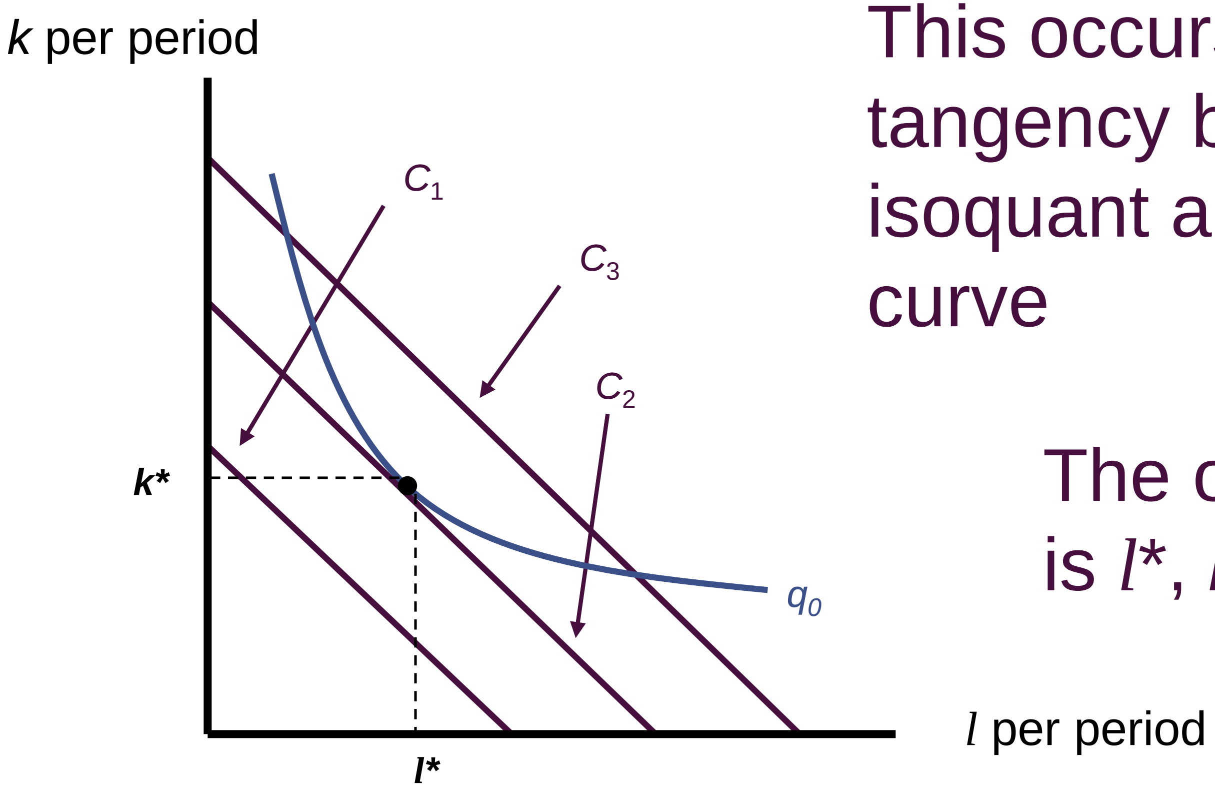



Costs are represented by parallel lines with a slope of - *w*/*v*

$C_1 < C_2 < C_3$

# Cost-Minimizing Input Choices

The minimum cost of producing $q_0$ is $C_2$




This occurs at the tangency between the isoquant and the total cost curve

The optimal choice is $l^*$, $k^*$

# Contingent Demand for Inputs

- In Chapter 4, we considered an individual's expenditure-minimization problem
  - we used this technique to develop the compensated demand for a good
- Can we develop a firm's demand for an input in the same way?

# Contingent Demand for Inputs

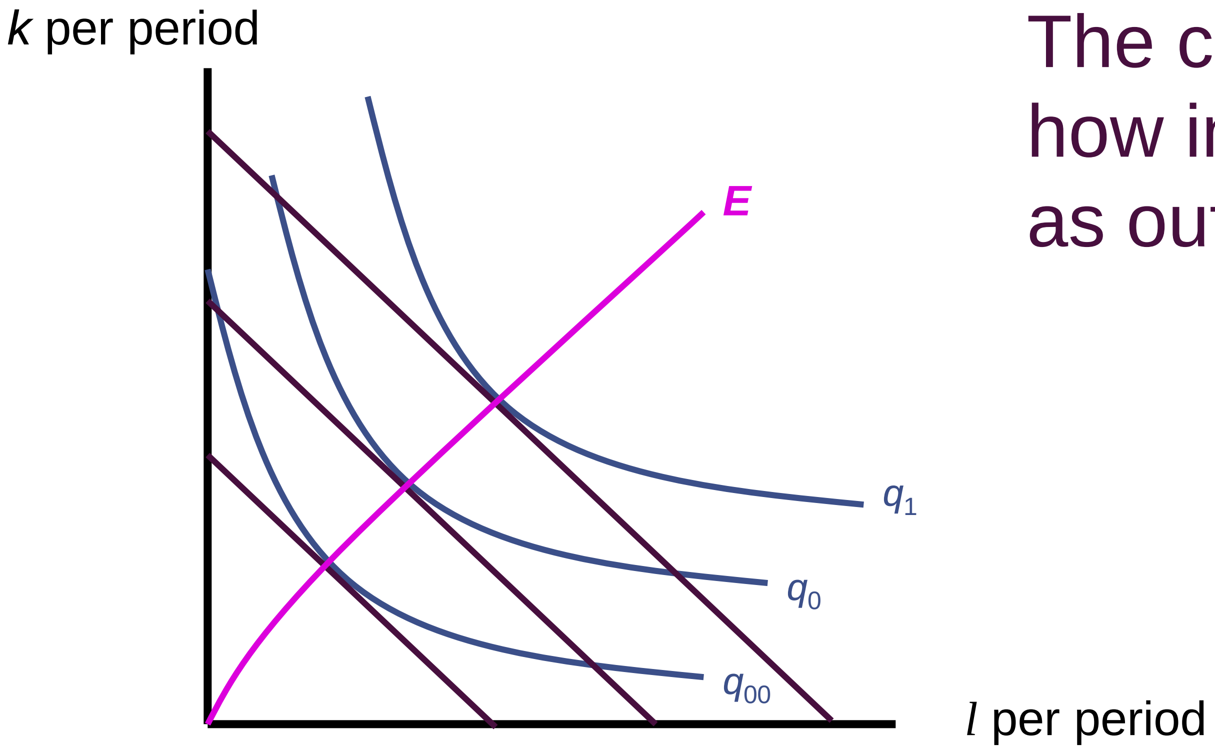
- In the present case, cost minimization leads to a demand for capital and labor that is contingent on the level of output being produced
- The demand for an input is a derived demand
  - it is based on the level of the firm's output

# The Firm's Expansion Path

- The firm can determine the cost-minimizing combinations of *k* and *l* for every level of output
- If input costs remain constant for all amounts of *k* and *l* the firm may demand, we can trace the locus of cost-minimizing choices
  – called the firm's <u>expansion path</u>

# The Firm’s Expansion Path

The expansion path is the locus of cost-minimizing tangencies




The curve shows how inputs increase as output increases

# The Firm's Expansion Path

- The expansion path does not have to be a straight line
  - the use of some inputs may increase faster than others as output expands
    - depends on the shape of the isoquants
- The expansion path does not have to be upward sloping
  - if the use of an input falls as output expands, that input is an <u>inferior input</u>

# Cost Minimization

- Suppose that the production function is Cobb-Douglas:

$$q = k^{\alpha} l^{\beta}$$

- The Lagrangian expression for cost minimization of producing $q_0$ is

$$\mathbf{L} = vk + wl + \lambda(q_0 - k^{\alpha} l^{\beta})$$

# Cost Minimization

- The first-order conditions for a minimum are

$$\partial \mathbf{L}/\partial k = v - \lambda\alpha k^{\alpha-1} l^{\beta} = 0$$

$$\partial \mathbf{L}/\partial l = w - \lambda\beta k^{\alpha} l^{\beta-1} = 0$$

$$\partial \mathbf{L}/\partial \lambda = q_0 - k^{\alpha} l^{\beta} = 0$$

# Cost Minimization

- Dividing the first equation by the second gives us

$$\frac{w}{v} = \frac{\beta k^{\alpha} l^{\beta-1}}{\alpha k^{\alpha-1} l^{\beta}} = \frac{\beta}{\alpha} \cdot \frac{k}{l} = RTS$$

- This production function is homothetic
  - the *RTS* depends only on the ratio of the two inputs
  - the expansion path is a straight line

# Cost Minimization

- Suppose that the production function is CES:

$$q = (k^{\rho} + l^{\rho})^{\gamma/\rho}$$

- The Lagrangian expression for cost minimization of producing $q_0$ is

$$\mathbf{L} = vk + wl + \lambda[q_0 - (k^{\rho} + l^{\rho})^{\gamma/\rho}]$$

# Cost Minimization

- The first-order conditions for a minimum are

$$\partial \mathbf{L}/\partial k = v - \lambda(\gamma/\rho)(k^\rho + l^\rho)^{(\gamma-\rho)/\rho}(\rho)k^{\rho-1} = 0$$

$$\partial \mathbf{L}/\partial l = w - \lambda(\gamma/\rho)(k^\rho + l^\rho)^{(\gamma-\rho)/\rho}(\rho)l^{\rho-1} = 0$$

$$\partial \mathbf{L}/\partial \lambda = q_0 - (k^\rho + l^\rho)^{\gamma/\rho} = 0$$

# Cost Minimization

- Dividing the first equation by the second gives us

$$\frac{w}{v} = \left(\frac{1}{k}\right)^{\rho-1} = \left(\frac{k}{l}\right)^{1-\rho} = \left(\frac{k}{l}\right)^{1/\sigma}$$

- This production function is also homothetic

# Total Cost Function

- The total cost function shows that for any set of input costs and for any output level, the minimum cost incurred by the firm is

$$C = C(v,w,q)$$

- As output ($q$) increases, total costs increase

# Average Cost Function

- The <u>average cost function</u> (*AC*) is found by computing total costs per unit of output

$$\text{average cost} = AC(v,w,q) = \frac{C(v,w,q)}{q}$$

# Marginal Cost Function

- The marginal cost function (*MC*) is found by computing the change in total costs for a change in output produced

$$\text{marginal cost} = MC(v,w,q) = \frac{\partial C(v,w,q)}{\partial q}$$

# Graphical Analysis of Total Costs

- Suppose that $k_1$ units of capital and $l_1$ units of labor input are required to produce one unit of output

$$C(q=1) = vk_1 + wl_1$$

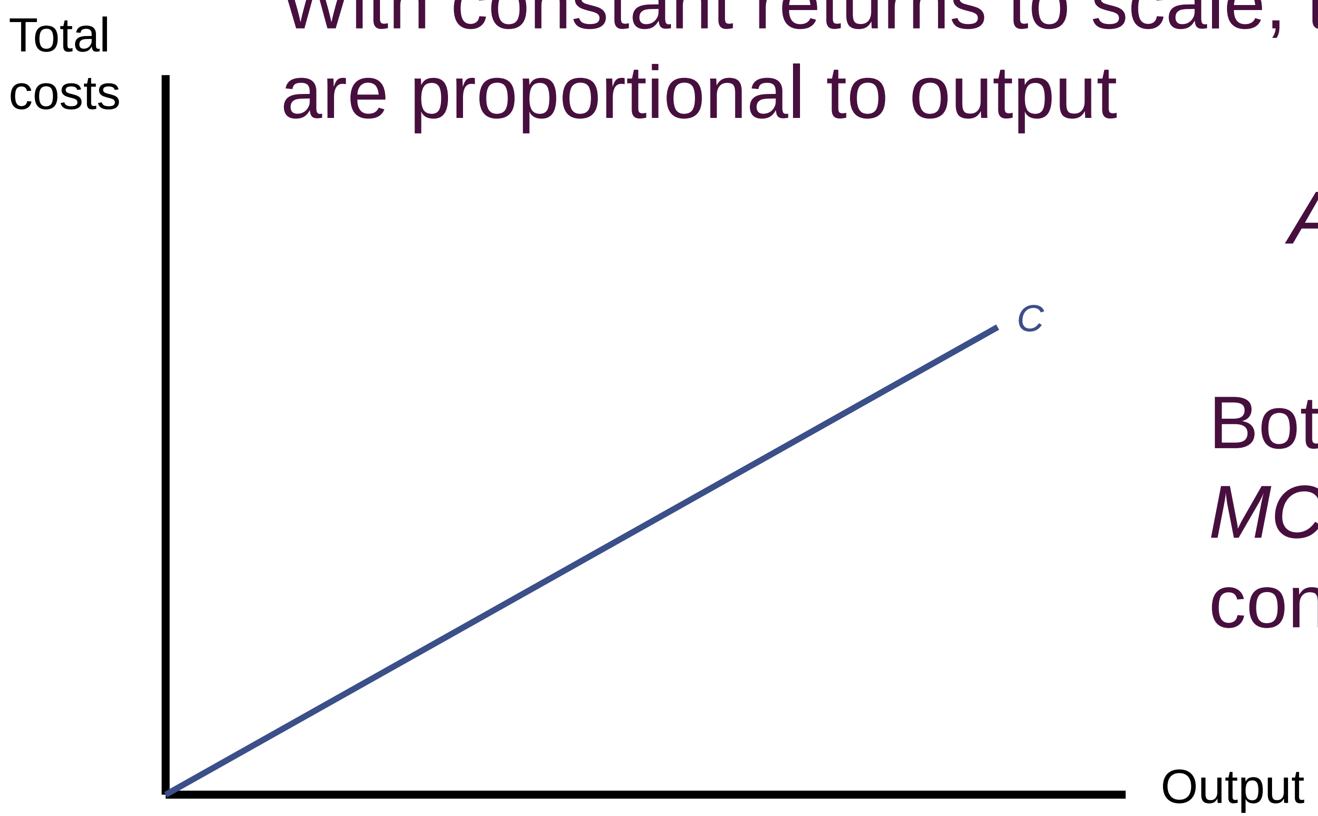
- To produce $m$ units of output (assuming constant returns to scale)

$$C(q=m) = vmk_1 + wml_1 = m(vk_1 + wl_1)$$

$$C(q=m) = m \cdot C(q=1)$$

# Graphical Analysis of Total Costs

With constant returns to scale, total costs are proportional to output



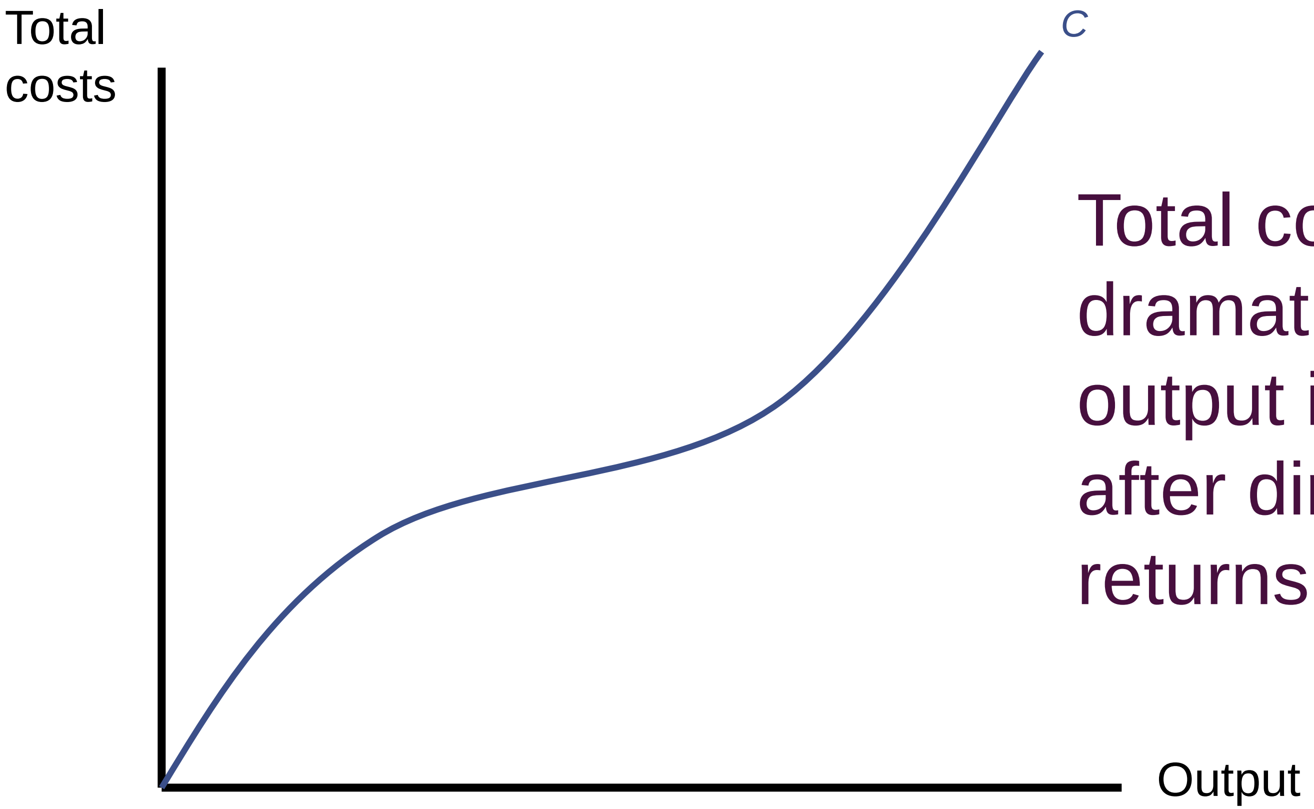

*AC* = *MC*

Both *AC* and *MC* will be constant

# Graphical Analysis of Total Costs

- Suppose instead that total costs start out as concave and then becomes convex as output increases
  - one possible explanation for this is that there is a third factor of production that is fixed as capital and labor usage expands
  - total costs begin rising rapidly after diminishing returns set in

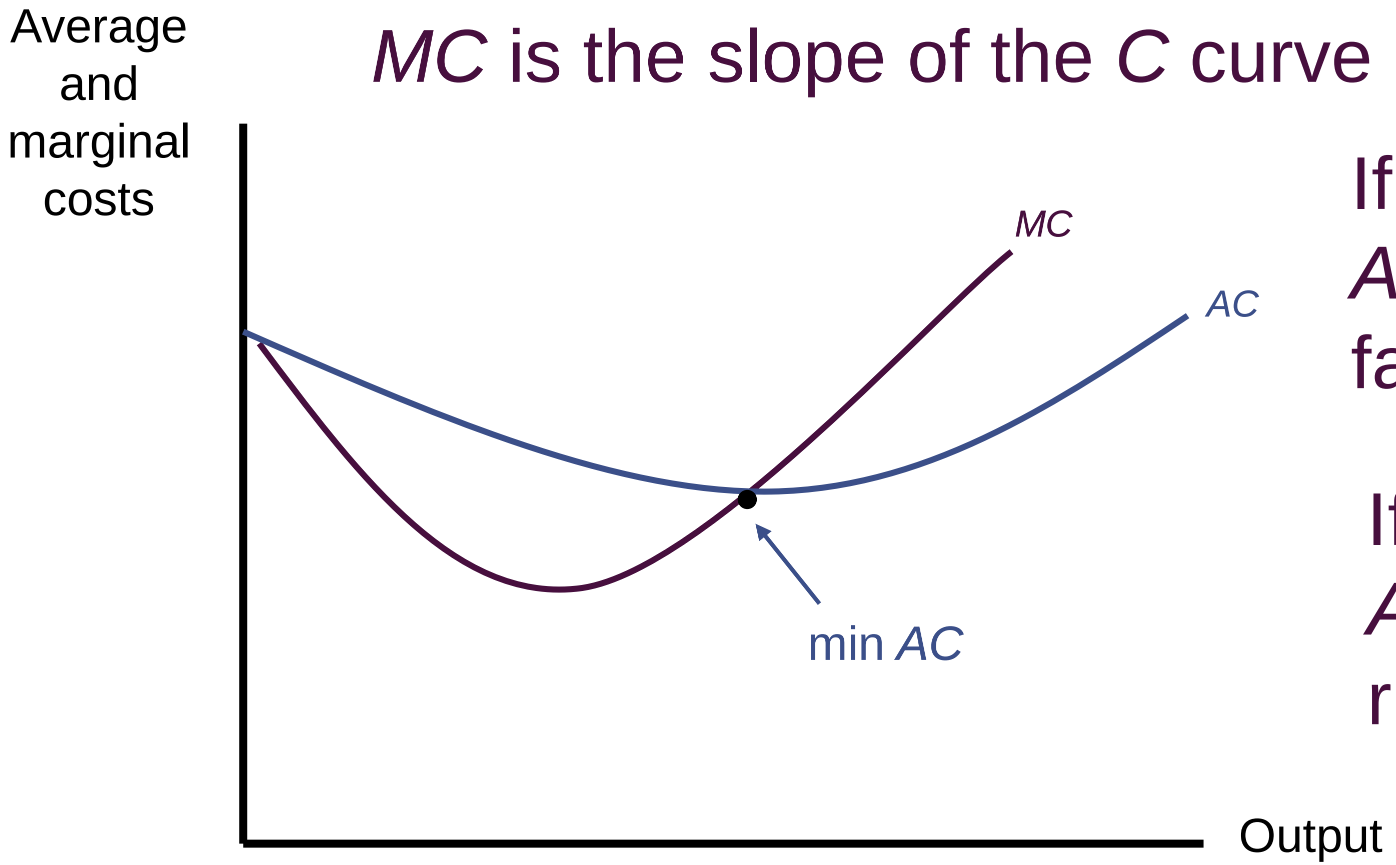
# Graphical Analysis of Total Costs




Total costs rise dramatically as output increases after diminishing returns set in

# Graphical Analysis of Total Costs

*MC* is the slope of the *C* curve




If *AC* > *MC*, *AC* must be falling

If *AC* < *MC*, *AC* must be rising

# Shifts in Cost Curves

- The cost curves are drawn under the assumption that input prices and the level of technology are held constant
  – any change in these factors will cause the cost curves to shift

# Some Illustrative Cost Functions

- Suppose we have a fixed proportions technology such that

$$q = f(k,l) = \min(ak,bl)$$

- Production will occur at the vertex of the L-shaped isoquants ($q = ak = bl$)

$$C(w,v,q) = vk + wl = v(q/a) + w(q/b)$$

$$C(w,v,q) = a\left(\frac{v}{a} + \frac{w}{b}\right)$$

# Some Illustrative Cost Functions

- Suppose we have a Cobb-Douglas technology such that

$$q = f(k,l) = k^{\alpha} l^{\beta}$$

- Cost minimization requires that

$$\frac{w}{v} = \frac{\beta}{\alpha} \cdot \frac{k}{l}$$

$$k = \frac{\alpha}{\beta} \cdot \frac{w}{v} \cdot l$$

# Some Illustrative Cost Functions

- If we substitute into the production function and solve for *l*, we will get

$$l = q^{1/\alpha+\beta}\left(\frac{\beta}{\alpha}\right)^{\alpha/\alpha+\beta} w^{-\alpha/\alpha+\beta} v^{\alpha/\alpha+\beta}$$

- A similar method will yield

$$k = q^{1/\alpha+\beta}\left(\frac{\alpha}{\beta}\right)^{\beta/\alpha+\beta} w^{\beta/\alpha+\beta} v^{-\beta/\alpha+\beta}$$

# Some Illustrative Cost Functions

- Now we can derive total costs as

$$C(v,w,q) = vk + wl = q^{1/\alpha+\beta} B v^{\alpha/\alpha+\beta} w^{\beta/\alpha+\beta}$$

where

$$B = (\alpha + \beta)\alpha^{-\alpha/\alpha+\beta}\beta^{-\beta/\alpha+\beta}$$

which is a constant that involves only the parameters $\alpha$ and $\beta$

# Some Illustrative Cost Functions

- Suppose we have a CES technology such that

$$q = f(k,l) = (k^{\rho} + l^{\rho})^{\gamma/\rho}$$

- To derive the total cost, we would use the same method and eventually get

$$C(v,w,q) = vk + wl = q^{1/\gamma}(v^{\rho/\rho-1} + w^{\rho/\rho-1})^{(\rho-1)/\rho}$$

$$C(v,w,q) = q^{1/\gamma}(v^{1-\sigma} + w^{1-\sigma})^{1/1-\sigma}$$

# Properties of Cost Functions

- Homogeneity
  - cost functions are all homogeneous of degree one in the input prices
    - cost minimization requires that the ratio of input prices be set equal to *RTS*, a doubling of all input prices will not change the levels of inputs purchased
    - pure, uniform inflation will not change a firm’s input decisions but will shift the cost curves up

# Properties of Cost Functions

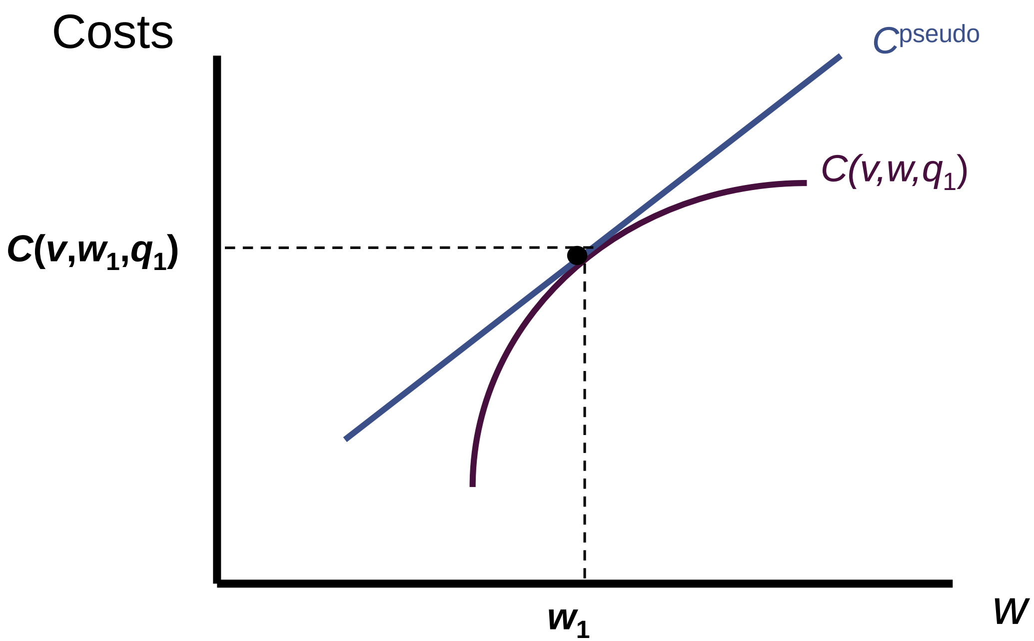
- Nondecreasing in *q*, *v*, and *w*
  - cost functions are derived from a cost-minimization process
    - any decline in costs from an increase in one of the function's arguments would lead to a contradiction

# Properties of Cost Functions

- Concave in input prices
  - costs will be lower when a firm faces input prices that fluctuate around a given level than when they remain constant at that level
    - the firm can adapt its input mix to take advantage of such fluctuations

# Concavity of Cost Function

At $w_1$, the firm's costs are $C(v,w_1,q_1)$




If the firm continues to buy the same input mix as $w$ changes, its cost function would be $C^{pseudo}$

Since the firm's input mix will likely change, actual costs will be less than $C^{pseudo}$ such as $C(v,w,q_1)$

# Properties of Cost Functions

- Some of these properties carry over to average and marginal costs
  - homogeneity
  - effects of $v$, $w$, and $q$ are ambiguous

# Input Substitution (skipped)

- A change in the price of an input will cause the firm to alter its input mix
- We wish to see how *k*/*l* changes in response to a change in *w*/*v*, while holding *q* constant

$$\frac{\partial\left(\frac{k}{l}\right)}{\partial\left(\frac{w}{v}\right)}$$

# Input Substitution

- Putting this in proportional terms as

$$s = \frac{\partial(k/l)}{\partial(w/v)} \cdot \frac{w/v}{k/l} = \frac{\partial \ln(k/l)}{\partial \ln(w/v)}$$

gives an alternative definition of the elasticity of substitution

  - in the two-input case, *s* must be nonnegative
  - large values of *s* indicate that firms change their input mix significantly if input prices change

# Partial Elasticity of Substitution

- The partial elasticity of substitution between two inputs ($x_i$ and $x_j$) with prices $w_i$ and $w_j$ is given by

$$s_{ij} = \frac{\partial(x_i / x_j)}{\partial(w_j / w_i)} \cdot \frac{w_j / w_i}{x_i / x_j} = \frac{\partial \ln(x_i / x_j)}{\partial \ln(w_j / w_i)}$$

- $S_{ij}$ is a more flexible concept than $\sigma$ because it allows the firm to alter the usage of inputs other than $x_i$ and $x_j$ when input prices change

# Size of Shifts in Costs Curves

- The increase in costs will be largely influenced by the relative significance of the input in the production process
- If firms can easily substitute another input for the one that has risen in price, there may be little increase in costs

# Technical Progress(skipped)

- Improvements in technology also lower cost curves
- Suppose that total costs (with constant returns to scale) are

$$C_0 = C_0(q,v,w) = qC_0(v,w,1)$$

# Technical Progress

- Because the same inputs that produced one unit of output in period zero will produce *A*(*t*) units in period *t*

$$C_t(v,w,A(t)) = A(t)C_t(v,w,1) = C_0(v,w,1)$$

- Total costs are given by

$$C_t(v,w,q) = qC_t(v,w,1) = qC_0(v,w,1)/A(t)$$

$$= C_0(v,w,q)/A(t)$$

# Shifting the Cobb-Douglas Cost Function

- The Cobb-Douglas cost function is

$$C(v,w,q) = vk + wl = q^{1/\alpha+\beta} B v^{\alpha/\alpha+\beta} w^{\beta/\alpha+\beta}$$

where

$$B = (\alpha+\beta)\alpha^{-\alpha/\alpha+\beta}\beta^{-\beta/\alpha+\beta}$$

- If we assume $\alpha = \beta = 0.5$, the total cost curve is greatly simplified:

$$C(v,w,q) = vk + wl = 2qv^{0.5}w^{0.5}$$

# Shifting the Cobb-Douglas Cost Function

- If $v$ = 3 and $w$ = 12, the relationship is

$$C(3,12,q) = 2q\sqrt{36} = 12q$$

  - $C$ = 480 to produce $q$ =40
  - $AC = C/q = 12$
  - $MC = \partial C/\partial q = 12$

# Shifting the Cobb-Douglas Cost Function

- If $v$ = 3 and $w$ = 27, the relationship is

$$C(3,27,q) = 2q\sqrt{81} = 18q$$

  - $C$ = 720 to produce $q$ =40
  - $AC = C/q = 18$
  - $MC = \partial C/\partial q = 18$

# Contingent Demand for Inputs(条件要素需求)

- Contingent demand functions for all of the firms inputs can be derived from the cost function
  - Shephard's lemma
    - the contingent demand function for any input is given by the partial derivative of the total-cost function with respect to that input's price

# Contingent Demand for Inputs

- Suppose we have a fixed proportions technology
- The cost function is

$$C(w,v,q) = q\left(\frac{v}{a} + \frac{w}{b}\right)$$

# Contingent Demand for Inputs

- For this cost function, contingent demand functions are quite simple:

$$k^c(v,w,q) = \frac{\partial C(v,w,q)}{\partial v} = \frac{q}{a}$$

$$l^c(v,w,q) = \frac{\partial C(v,w,q)}{\partial w} = \frac{q}{b}$$

# Contingent Demand for Inputs

- Suppose we have a Cobb-Douglas technology
- The cost function is

$$C(v,w,q) = vk + wl = q^{1/\alpha+\beta} B v^{\alpha/\alpha+\beta} w^{\beta/\alpha+\beta}$$

# Contingent Demand for Inputs

- For this cost function, the derivation is messier:

$$k^c(v,w,q) = \frac{\partial C}{\partial v} = \frac{\alpha}{\alpha+\beta} \cdot q^{1/\alpha+\beta} B v^{-\beta/\alpha+\beta} w^{\beta/\alpha+\beta}$$

$$= \frac{\alpha}{\alpha+\beta} \cdot q^{1/\alpha+\beta} B \left(\frac{w}{v}\right)^{\beta/\alpha+\beta}$$

# Contingent Demand for Inputs

$$l^c(v,w,q) = \frac{\partial C}{\partial w} = \frac{\beta}{\alpha+\beta} \cdot q^{1/\alpha+\beta} B v^{\alpha/\alpha+\beta} w^{-\alpha/\alpha+\beta}$$

$$= \frac{\beta}{\alpha+\beta} \cdot q^{1/\alpha+\beta} B \left(\frac{w}{v}\right)^{-\alpha/\alpha+\beta}$$

- The contingent demands for inputs depend on both inputs' prices

# Short-Run, Long-Run Distinction

- In the short run, economic actors have only limited flexibility in their actions
- Assume that the capital input is held constant at $k_1$ and the firm is free to vary only its labor input
- The production function becomes

$$q = f(k_1, l)$$

# Short-Run Total Costs

- Short-run total cost for the firm is

$$SC = vk_1 + wl$$

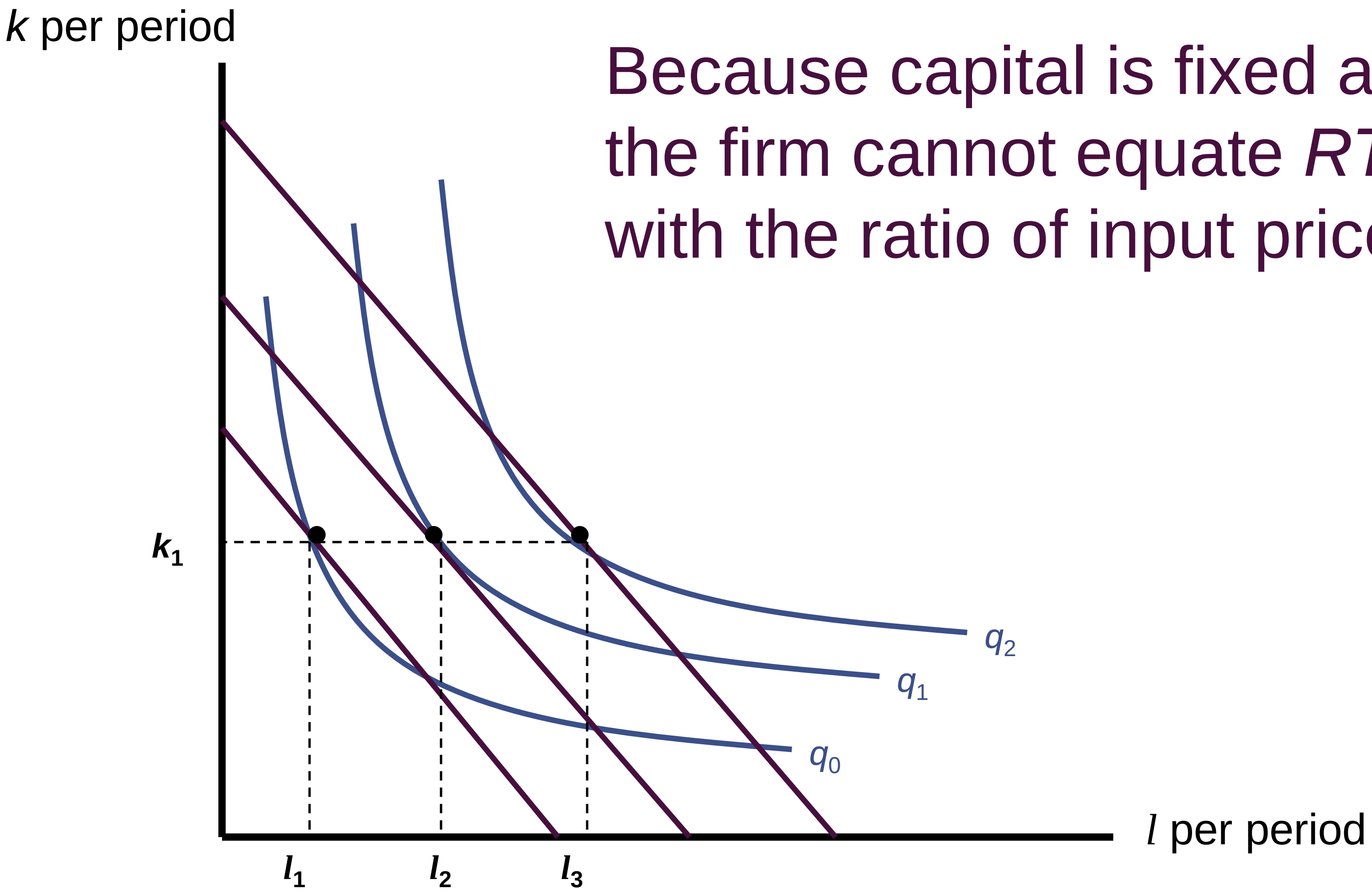
- There are two types of short-run costs:
  - short-run fixed costs are costs associated with fixed inputs ($vk_1$)
  - short-run variable costs are costs associated with variable inputs ($wl$)

# Short-Run Total Costs

- Short-run costs are not minimal costs for producing the various output levels
  - the firm does not have the flexibility of input choice
  - to vary its output in the short run, the firm must use nonoptimal input combinations
  - the *RTS* will not be equal to the ratio of input prices

# Short-Run Total Costs

Because capital is fixed at $k_1$, the firm cannot equate *RTS* with the ratio of input prices



# Short-Run Marginal and Average Costs

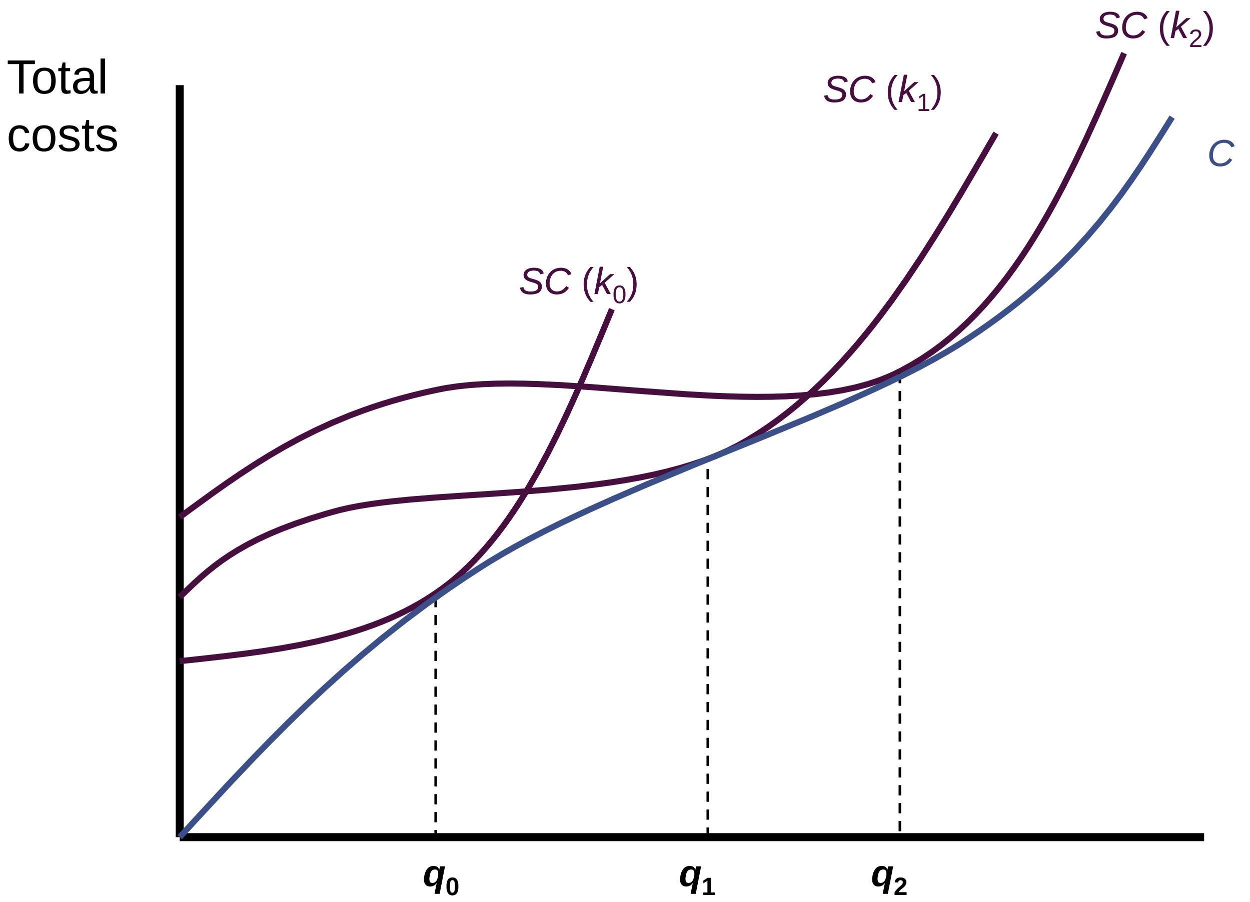
- The short-run average total cost (*SAC*) function is

$$SAC = \text{total costs/total output} = SC/q$$

- The short-run marginal cost (*SMC*) function is

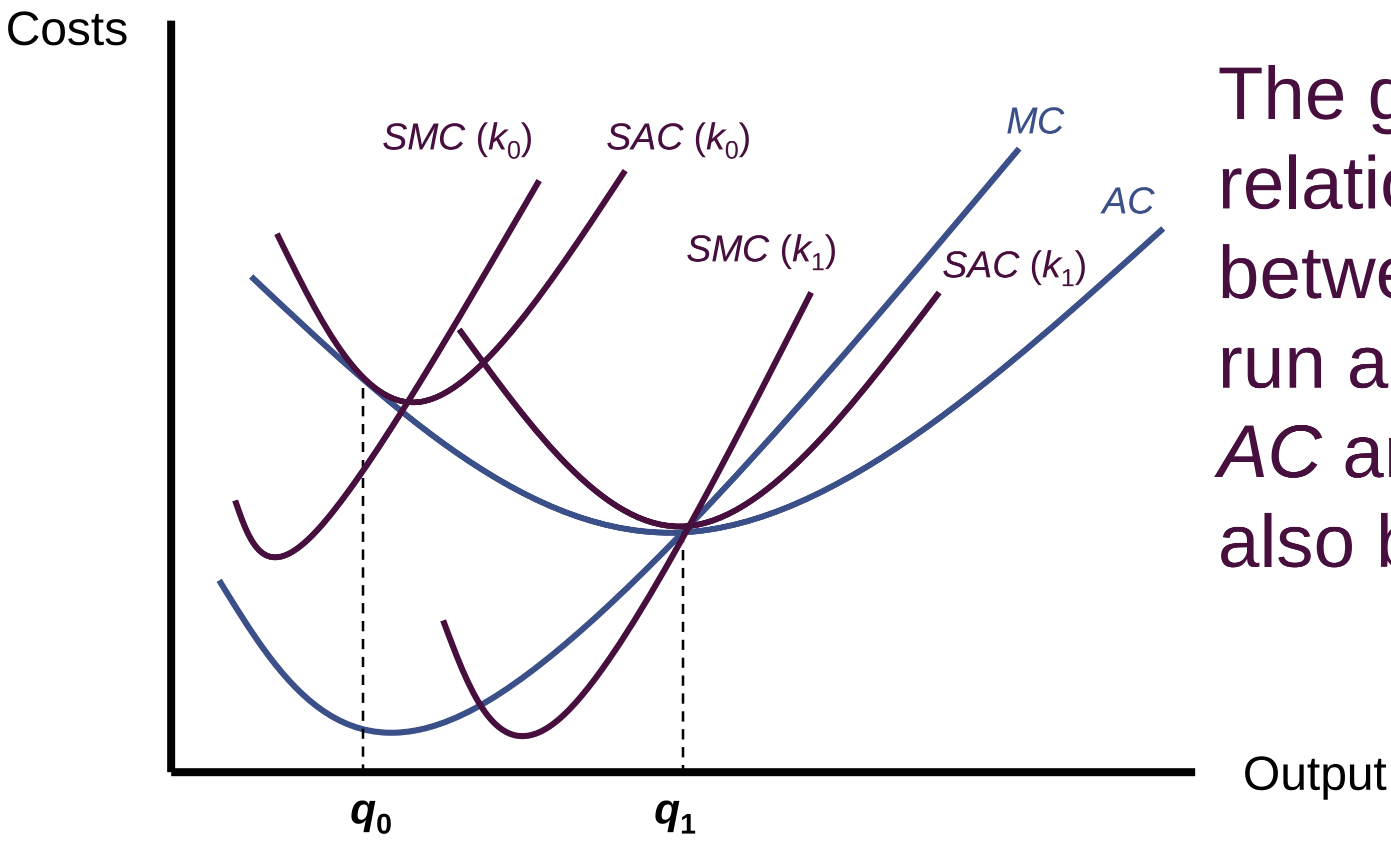
$$SMC = \text{change in } SC/\text{change in output} = \partial SC/\partial q$$

# Relationship between Short-Run and Long-Run Costs




The long-run $C$ curve can be derived by varying the level of $k$

# Relationship between Short-Run and Long-Run Costs




The geometric relationship between short-run and long-run *AC* and *MC* can also be shown

# Relationship between Short-Run and Long-Run Costs

- At the minimum point of the *AC* curve:
  - the *MC* curve crosses the *AC* curve
    - *MC* = *AC* at this point
  - the *SAC* curve is tangent to the *AC* curve
    - *SAC* (for this level of *k*) is minimized at the same level of output as *AC*
    - *SMC* intersects *SAC* also at this point

$$AC = MC = SAC = SMC$$

# Important Points to Note:

- A firm that wishes to minimize the economic costs of producing a particular level of output should choose that input combination for which the rate of technical substitution (*RTS*) is equal to the ratio of the inputs' rental prices

# Important Points to Note:

- Repeated application of this minimization procedure yields the firm's expansion path
  - the expansion path shows how input usage expands with the level of output
    - it also shows the relationship between output level and total cost
    - this relationship is summarized by the total cost function, $C(v,w,q)$

# Important Points to Note:

- The firm's average cost ($AC = C/q$) and marginal cost ($MC = \partial C/\partial q$) can be derived directly from the total-cost function
  - if the total cost curve has a general cubic shape, the *AC* and *MC* curves will be u-shaped

# Important Points to Note:

- All cost curves are drawn on the assumption that the input prices are held constant
  - when an input price changes, cost curves shift to new positions
    - the size of the shifts will be determined by the overall importance of the input and the substitution abilities of the firm
  - technical progress will also shift cost curves

# Important Points to Note:

- Input demand functions can be derived from the firm's total-cost function through partial differentiation
  - these input demands will depend on the quantity of output the firm chooses to produce
    - are called "contingent" demand functions

# Important Points to Note:

- In the short run, the firm may not be able to vary some inputs
  - it can then alter its level of production only by changing the employment of its variable inputs
  - it may have to use nonoptimal, higher-cost input combinations than it would choose if it were possible to vary all inputs