Chapter 9

PROFIT MAXIMIZATION

The Nature of Firms

- A firm is an association of individuals who have organized themselves for the purpose of turning inputs into outputs
- Different individuals will provide different types of inputs
  - the nature of the contractual relationship between the providers of inputs to a firm may be quite complicated

Contractual Relationships

- Some contracts between providers of inputs may be explicit
  - may specify hours, work details, or compensation
- Other arrangements will be more implicit in nature
  - decision-making authority or sharing of tasks

Modeling Firms' Behavior

- Most economists treat the firm as a single decision-making unit
  - the decisions are made by a single dictatorial manager who rationally pursues some goal
    - usually profit-maximization

Profit Maximization

- A profit-maximizing firm chooses both its inputs and its outputs with the sole goal of achieving maximum economic profits
  - – seeks to maximize the difference between total revenue and total economic costs

Profit Maximization

- If firms are strictly profit maximizers, they will make decisions in a “marginal” way
  - examine the marginal profit obtainable from producing one more unit of hiring one additional laborer

Output Choice

- Total revenue for a firm is given by

$$R(q) = p(q) \cdot q$$

- In the production of *q*, certain economic costs are incurred [$C(q)$]
- Economic profits (π) are the difference between total revenue and total costs

$$\pi(q) = R(q) - C(q) = p(q) \cdot q - C(q)$$

Output Choice

- The necessary condition for choosing the level of q that maximizes profits can be found by setting the derivative of the π function with respect to q equal to zero

$$\frac{d\pi}{dq} = \pi'(q) = \frac{dR}{dq} - \frac{dC}{dq} = 0$$

$$\frac{dR}{dq} = \frac{dC}{dq}$$

Output Choice

- To maximize economic profits, the firm should choose the output for which marginal revenue is equal to marginal cost

$$MR = \frac{dR}{dq} = \frac{dC}{dq} = MC$$

Second-Order Conditions

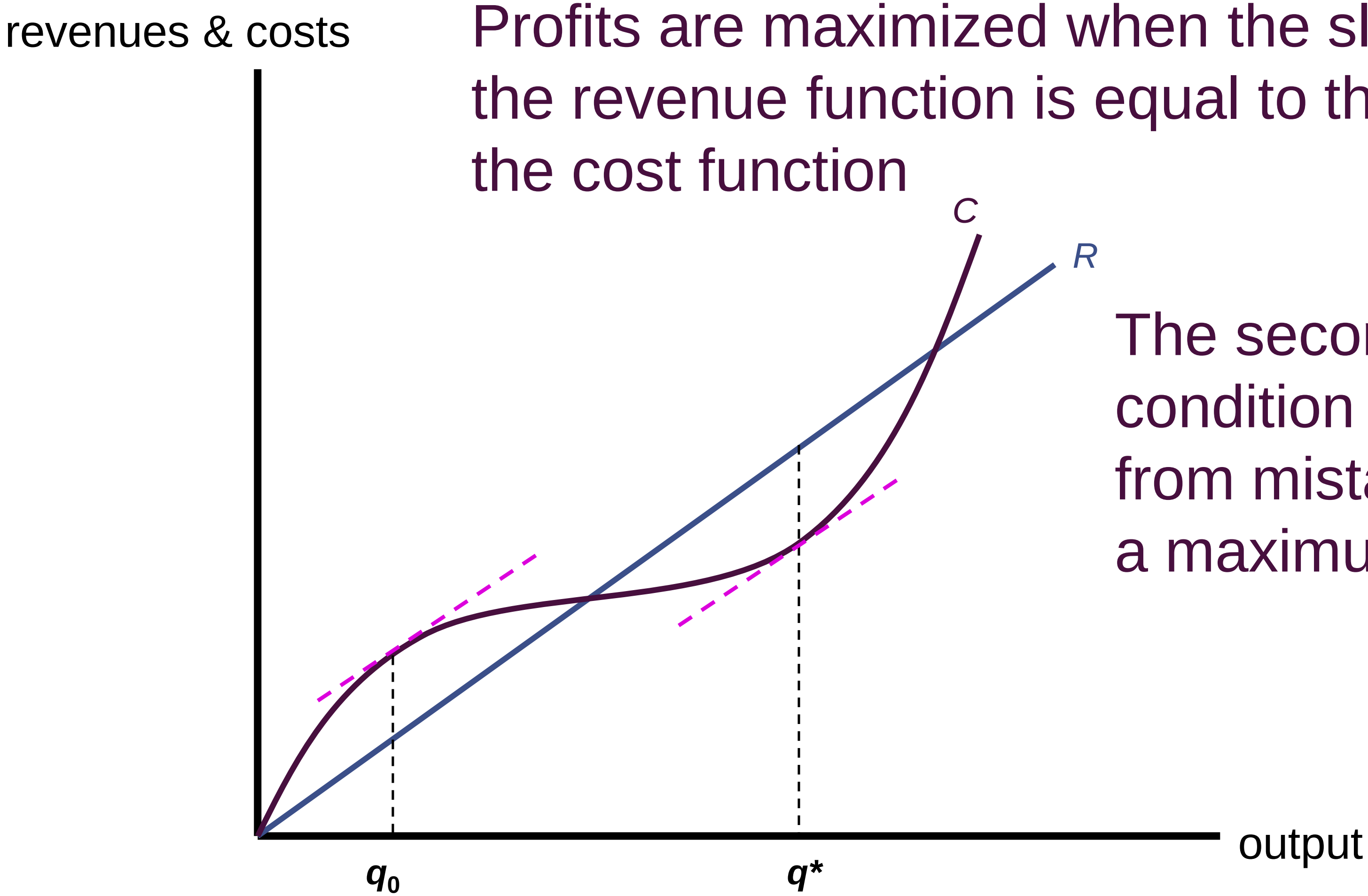
- *MR* = *MC* is only a necessary condition for profit maximization
- For sufficiency, it is also required that

$$\left.\frac{d^2\pi}{dq^2}\right|_{q=q^*} = \left.\frac{d\pi'(q)}{dq}\right|_{q=q^*} < 0$$

- "marginal" profit must be decreasing at the optimal level of *q*

Profit Maximization

Profits are maximized when the slope of the revenue function is equal to the slope of the cost function



The second-order condition prevents us from mistaking q_0 as a maximum

Marginal Revenue

- If a firm can sell all it wishes without having any effect on market price, marginal revenue will be equal to price
- If a firm faces a downward-sloping demand curve, more output can only be sold if the firm reduces the good's price

$$\text{marginal revenue} = MR(q) = \frac{dR}{dq} = \frac{d[p(q)\cdot q]}{dq} = p + q\cdot\frac{dp}{dq}$$

Marginal Revenue

- If a firm faces a downward-sloping demand curve, marginal revenue will be a function of output
- If price falls as a firm increases output, marginal revenue will be less than price

Marginal Revenue

- Suppose that the demand curve for a sub sandwich is

$$q = 100 - 10p$$

- Solving for price, we get

$$p = -q/10 + 10$$

- This means that total revenue is

$$R = pq = -q^2/10 + 10q$$

- Marginal revenue will be given by

$$MR = dR/dq = -q/5 + 10$$

Profit Maximization

- To determine the profit-maximizing output, we must know the firm's costs
- If subs can be produced at a constant average and marginal cost of $4, then

$$MR = MC$$

$$-q/5 + 10 = 4$$

$$q = 30$$

Marginal Revenue and Elasticity

- The concept of marginal revenue is directly related to the elasticity of the demand curve facing the firm
- The price elasticity of demand is equal to the percentage change in quantity that results from a one percent change in price

$$e_{q,p} = \frac{dq/q}{dp/p} = \frac{dq}{dp} \cdot \frac{p}{q}$$

Marginal Revenue and Elasticity

- This means that

$$MR = p + \frac{q \cdot dp}{dq} = p\left(1 + \frac{q}{p} \cdot \frac{dp}{dq}\right) = p\left(1 + \frac{1}{e_{q,p}}\right)$$

  – if the demand curve slopes downward, $e_{q,p} < 0$ and $MR < p$

  – if the demand is elastic, $e_{q,p} < -1$ and marginal revenue will be positive

    - if the demand is infinitely elastic, $e_{q,p} = -\infty$ and marginal revenue will equal price

Marginal Revenue and Elasticity

$e_{q,p} < -1$	$MR > 0$
$e_{q,p} = -1$	$MR = 0$
$e_{q,p} > -1$	$MR < 0$

The Inverse Elasticity Rule

- Because $MR = MC$ when the firm maximizes profit, we can see that

$$MC = p\left(1 + \frac{1}{e_{q,p}}\right) \qquad \frac{p - MC}{p} = -\frac{1}{e_{q,p}}$$

- The gap between price and marginal cost will fall as the demand curve facing the firm becomes more elastic

The Inverse Elasticity Rule

$$\frac{p - MC}{p} = -\frac{1}{e_{q,p}}$$

- If $e_{q,p} > -1$, $MC < 0$
- This means that firms will choose to operate only at points on the demand curve where demand is elastic

Average Revenue Curve

- If we assume that the firm must sell all its output at one price, we can think of the demand curve facing the firm as its average revenue curve
  - shows the revenue per unit yielded by alternative output choices

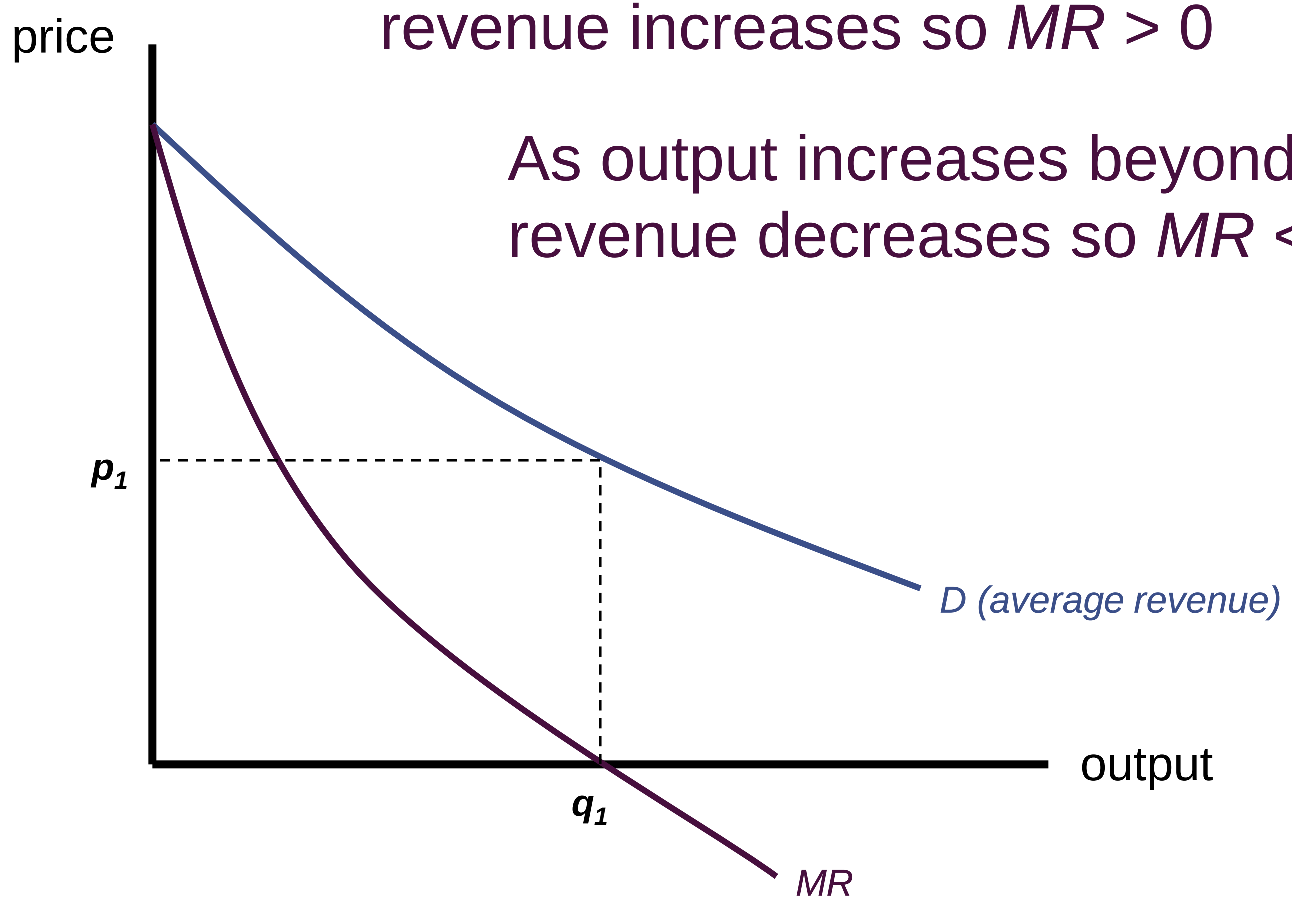
Marginal Revenue Curve

- The marginal revenue curve shows the extra revenue provided by the last unit sold
- In the case of a downward-sloping demand curve, the marginal revenue curve will lie below the demand curve

Marginal Revenue Curve

As output increases from 0 to q_1, total revenue increases so $MR > 0$

As output increases beyond q_1, total revenue decreases so $MR < 0$



Marginal Revenue Curve

- When the demand curve shifts, its associated marginal revenue curve shifts as well
  - a marginal revenue curve cannot be calculated without referring to a specific demand curve

The Constant Elasticity Case

- We showed (in Chapter 5) that a demand function of the form

$$q = ap^b$$

has a constant price elasticity of demand equal to *b*

- Solving this equation for *p*, we get

$$p = (1/a)^{1/b} q^{1/b} = kq^{1/b} \quad \text{where } k = (1/a)^{1/b}$$

The Constant Elasticity Case

- This means that

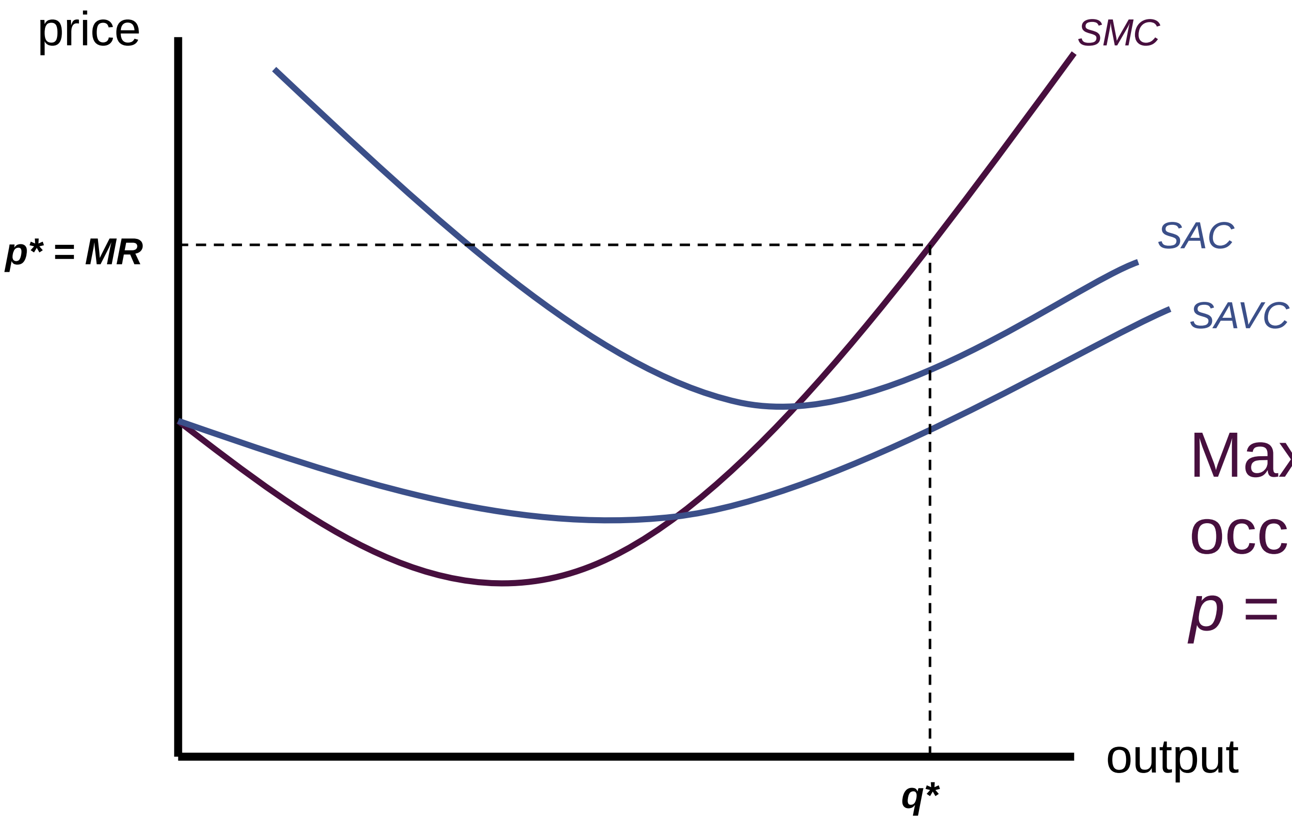
$$R = pq = kq^{(1+b)/b}$$

and

$$MR = dr/dq = [(1+b)/b]kq^{1/b} = [(1+b)/b]p$$

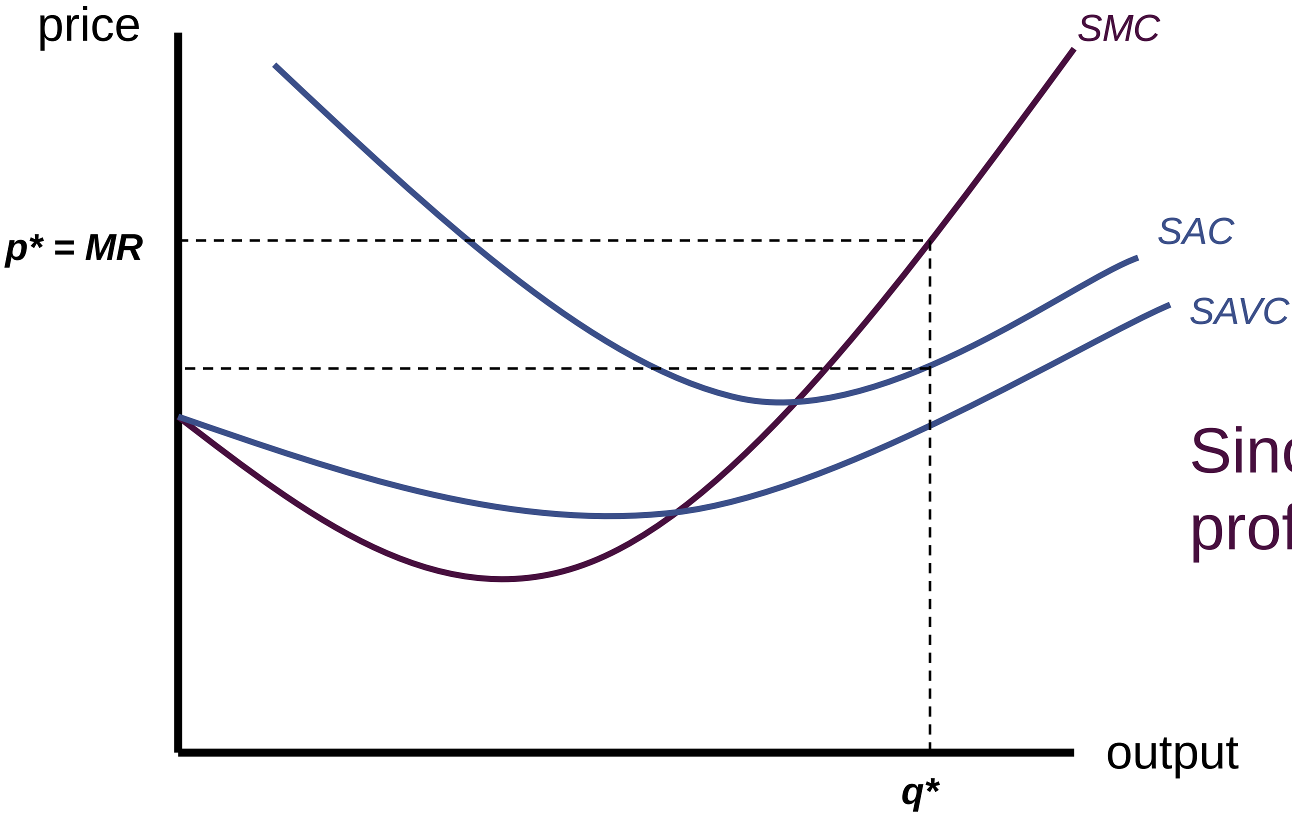
- This implies that *MR* is proportional to price

Short-Run Supply by a Price-Taking Firm



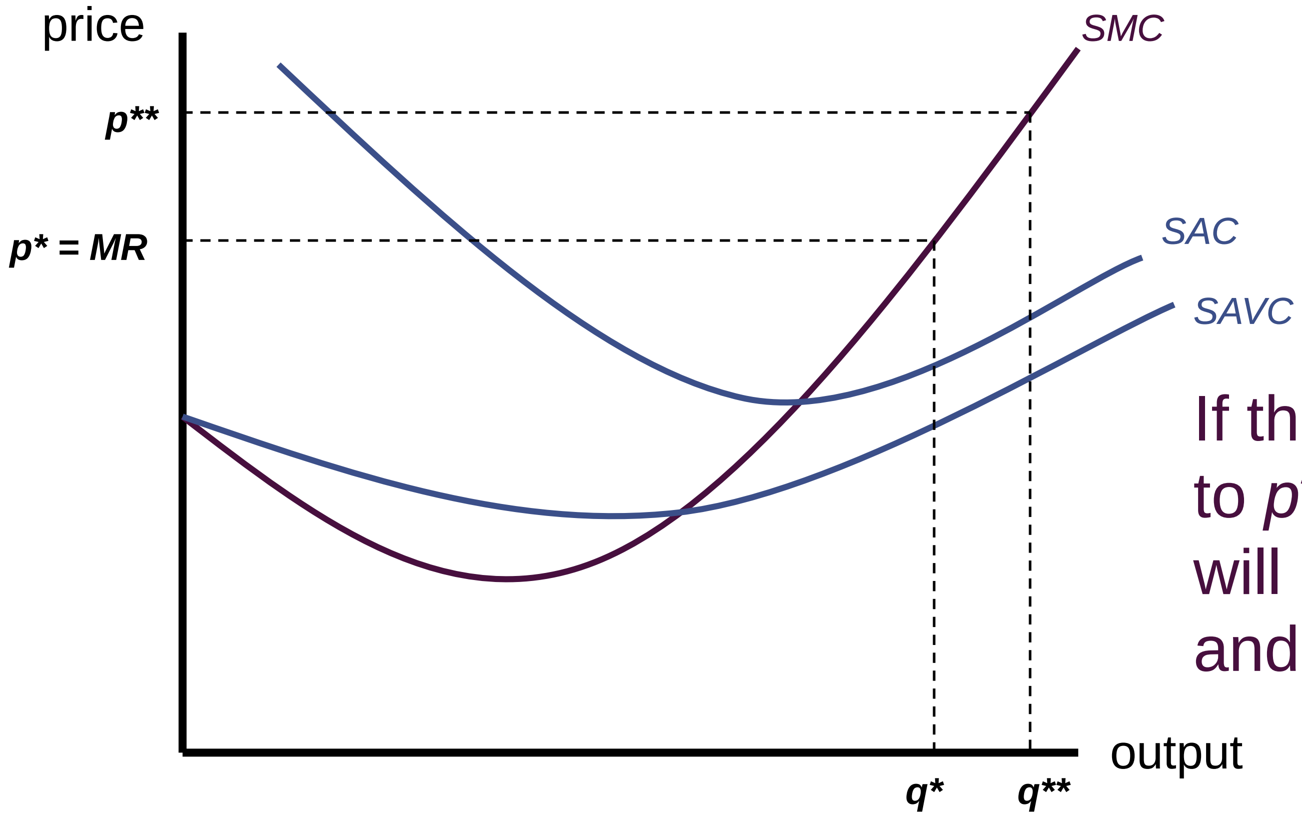
Maximum profit occurs where $p = SMC$

Short-Run Supply by a Price-Taking Firm



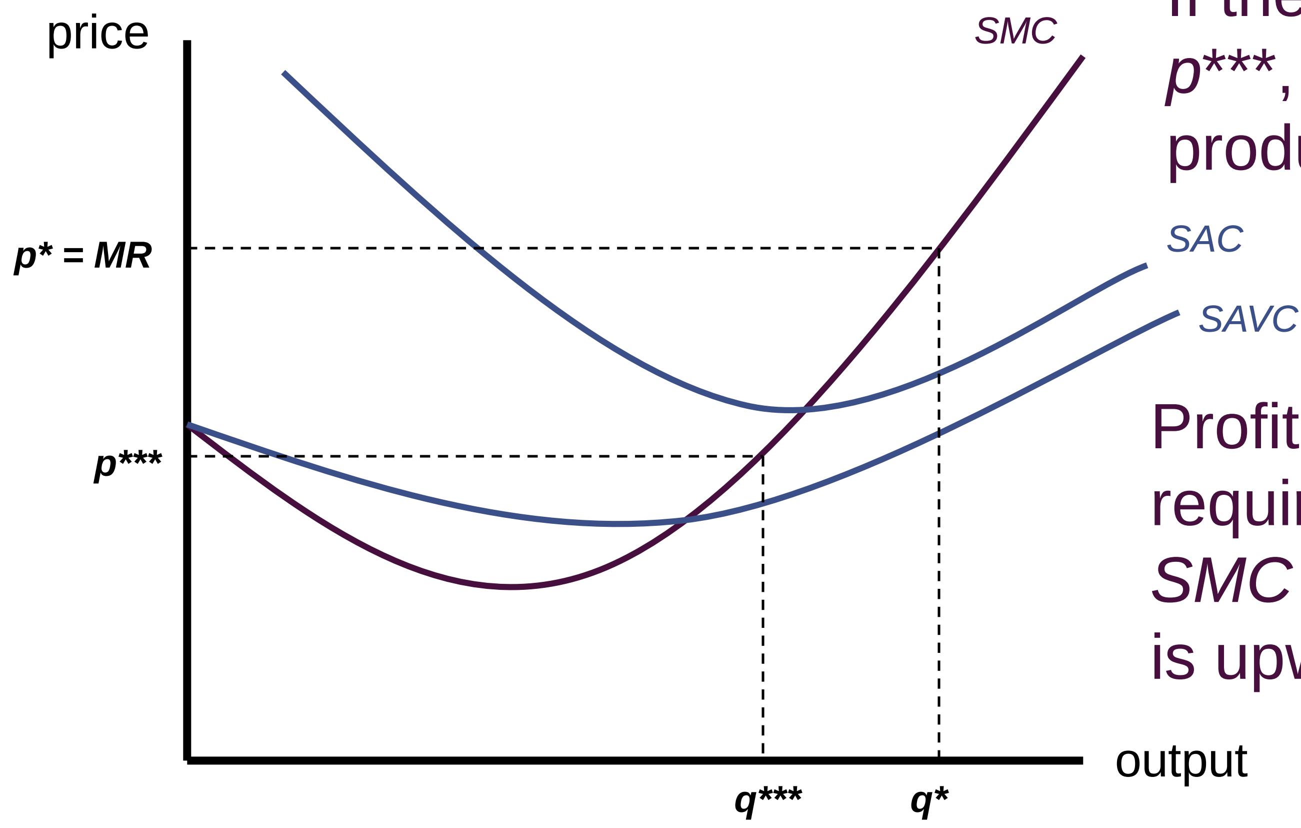
Since p > *SAC*, profit > 0

Short-Run Supply by a Price-Taking Firm



If the price rises to p^{**}, the firm will produce q^{**} and $\pi > 0$

Short-Run Supply by a Price-Taking Firm



If the price falls to *p****, the firm will produce *q****

Profit maximization requires that *p* = *SMC* and that *SMC* is upward-sloping

$\pi < 0$

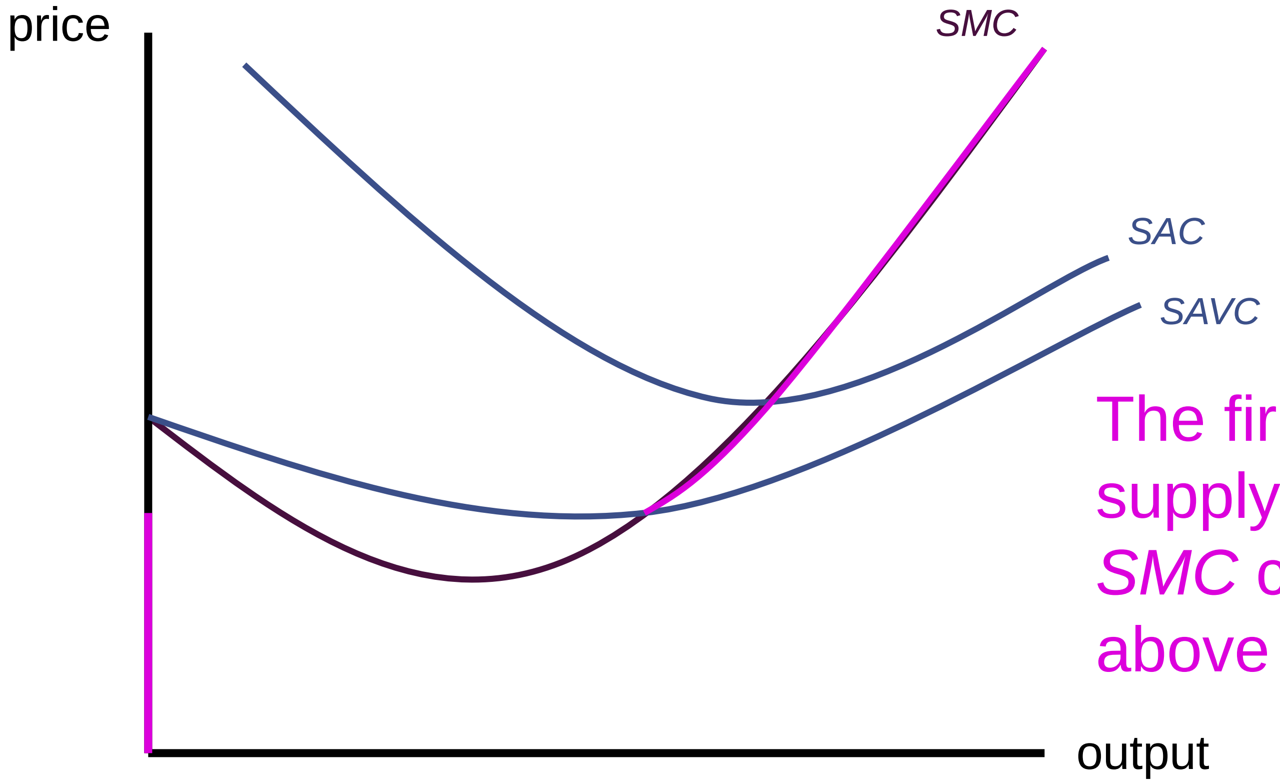
Short-Run Supply by a Price-Taking Firm

- The positively-sloped portion of the short-run marginal cost curve is the **short-run supply curve** for a price-taking firm
  - it shows how much the firm will produce at every possible market price
  - firms will only operate in the short run as long as total revenue covers variable cost
    - the firm will produce no output if $p < SAVC$

Short-Run Supply by a Price-Taking Firm

- Thus, the price-taking firm's short-run supply curve is the positively-sloped portion of the firm's short-run marginal cost curve above the point of minimum average variable cost
  - for prices below this level, the firm's profit-maximizing decision is to shut down and produce no output

Short-Run Supply by a Price-Taking Firm



The firm's short-run supply curve is the *SMC* curve that is above *SAVC*

Short-Run Supply

- Suppose that the firm's short-run total cost curve is

$$SC(v,w,q,k) = vk_1 + wq^{1/\beta}k_1^{-\alpha/\beta}$$

where k_1 is the level of capital held constant in the short run

- Short-run marginal cost is

$$SMC(v,w,q,k_1) = \frac{\partial SC}{\partial q} = \frac{w}{\beta} q^{(1-\beta)/\beta} k_1^{-\alpha/\beta}$$

Short-Run Supply

- The price-taking firm will maximize profit where $p = SMC$

$$SMC = \frac{w}{\beta} q^{(1-\beta)/\beta} k_1^{-\alpha/\beta} = p$$

- Therefore, quantity supplied will be

$$q = \left(\frac{w}{\beta}\right)^{-\beta/(1-\beta)} k_1^{-\alpha/(1-\beta)} p^{\beta/(1-\beta)}$$

Short-Run Supply

- To find the firm's shut-down price, we need to solve for *SAVC*

$$SVC = wq^{1/\beta}k_1^{-\alpha/\beta}$$

$$SAVC = SVC/q = wq^{(1-\beta)/\beta}k_1^{-\alpha/\beta}$$

- *SAVC* < *SMC* for all values of $\beta < 1$
  - there is no price low enough that the firm will want to shut down

Profit Functions

- A firm's economic profit can be expressed as a function of inputs

$$\pi = pq - C(q) = pf(k,l) - vk - wl$$

- Only the variables *k* and *l* are under the firm's control
  - the firm chooses levels of these inputs in order to maximize profits
    - treats *p*, *v*, and *w* as fixed parameters in its decisions

Profit Functions

- A firm's <u>profit function</u> shows its maximal profits as a function of the prices that the firm faces

$$\Pi(p,v,w) = \underset{k,l}{Max}\,\pi(k,l) = \underset{k,l}{Max}[pf(k,l) - vk - wl]$$

Properties of the Profit Function

- Homogeneity
  - the profit function is homogeneous of degree one in all prices
    - with pure inflation, a firm will not change its production plans and its level of profits will keep up with that inflation

Properties of the Profit Function

- Nondecreasing in output price
  - a firm could always respond to a rise in the price of its output by not changing its input or output plans
    - profits must rise

Properties of the Profit Function

- Nonincreasing in input prices
  - if the firm responded to an increase in an input price by not changing the level of that input, its costs would rise
    - profits would fall

Properties of the Profit Function

- Convex in output prices
  - the profits obtainable by averaging those from two different output prices will be at least as large as those obtainable from the average of the two prices

$$\frac{\Pi(p_1,v,w)+\Pi(p_2,v,w)}{2} \geq \Pi\left[\frac{p_1+p_2}{2},v,w\right]$$

Envelope Results

- We can apply the envelope theorem to see how profits respond to changes in output and input prices

$$\frac{\partial \Pi(p,v,w)}{\partial p} = q(p,v,w)$$

$$\frac{\partial \Pi(p,v,w)}{\partial v} = -k(p,v,w)$$

$$\frac{\partial \Pi(p,v,w)}{\partial w} = -l(p,v,w)$$

Producer Surplus in the Short Run

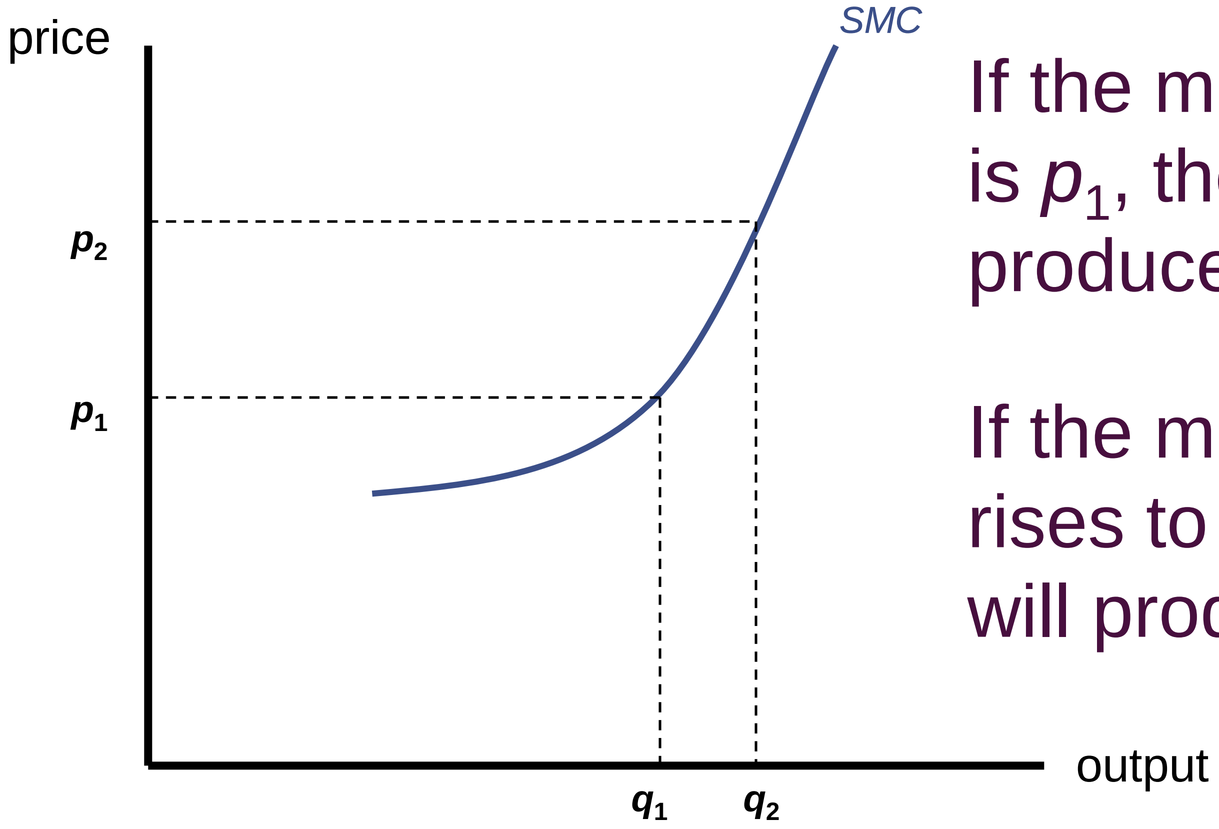
- Because the profit function is nondecreasing in output prices, we know that if $p_2 > p_1$

$$\Pi(p_2,\ldots) \geq \Pi(p_1,\ldots)$$

- The welfare gain to the firm of this price increase can be measured by

$$\text{welfare gain} = \Pi(p_2,\ldots) - \Pi(p_1,\ldots)$$

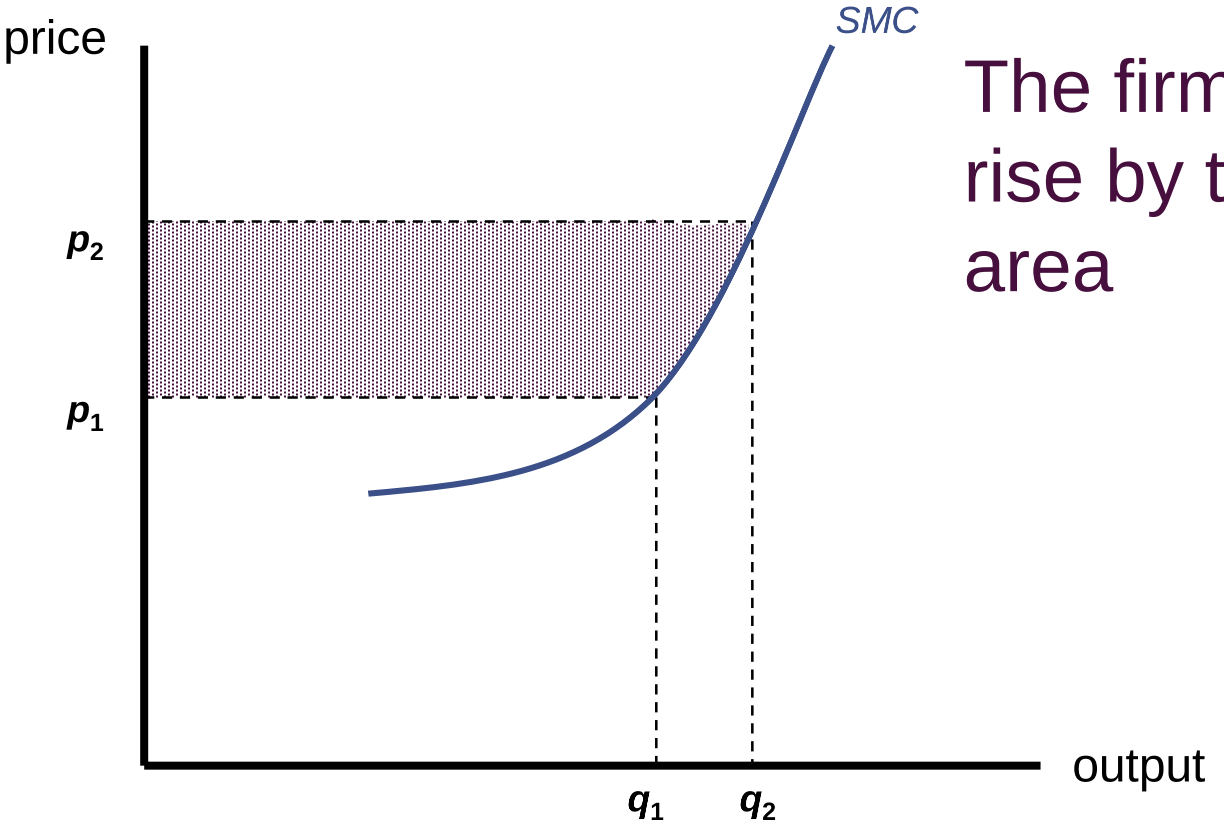
Producer Surplus in the Short Run



If the market price is p_1, the firm will produce q_1

If the market price rises to p_2, the firm will produce q_2

Producer Surplus in the Short Run



The firm's profits rise by the shaded area

Producer Surplus in the Short Run

- Mathematically, we can use the envelope theorem results

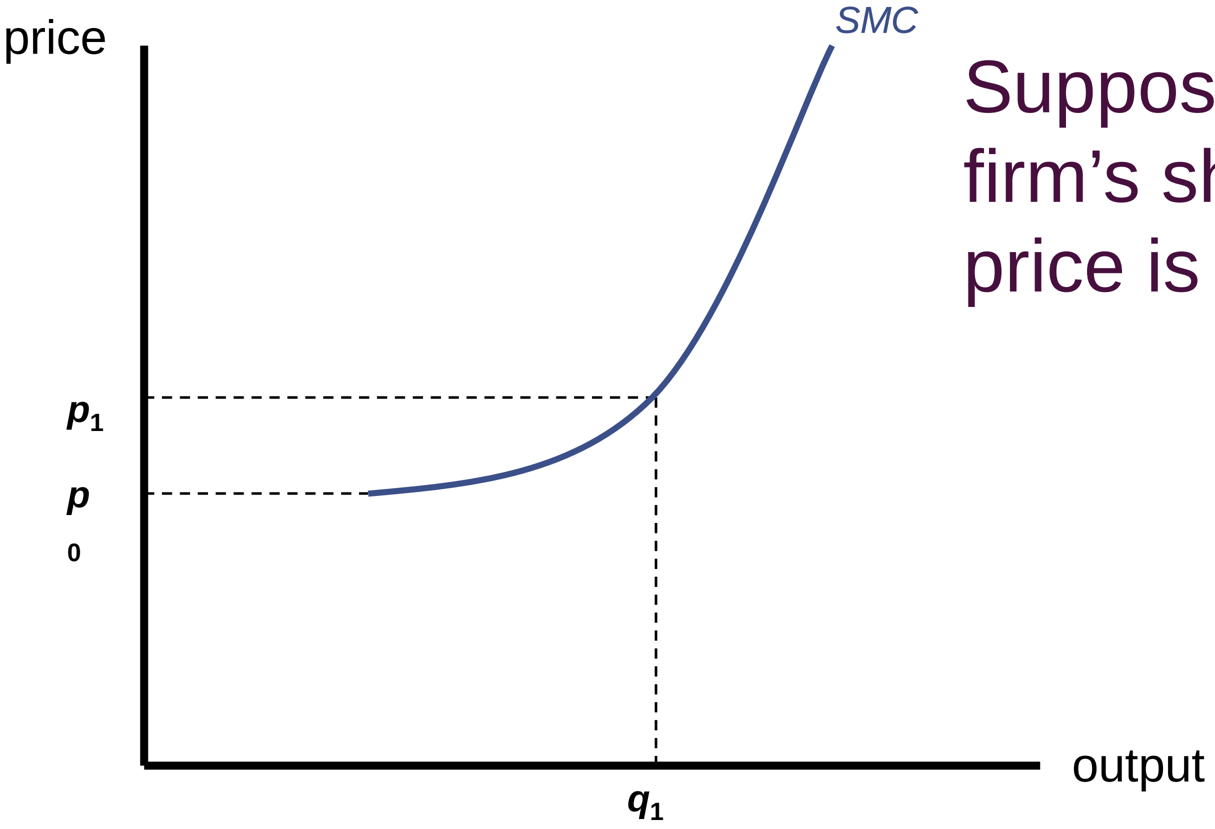
$$\text{welfare gain} = \int_{p_1}^{p_2} q(p)dp = \int_{p_1}^{p_2} (\partial \Pi / \partial p)dp$$

$$= \Pi(p_2, \ldots) - \Pi(p_1, \ldots)$$

Producer Surplus in the Short Run

- We can measure how much the firm values the right to produce at the prevailing price relative to a situation where it would produce no output

Producer Surplus in the Short Run



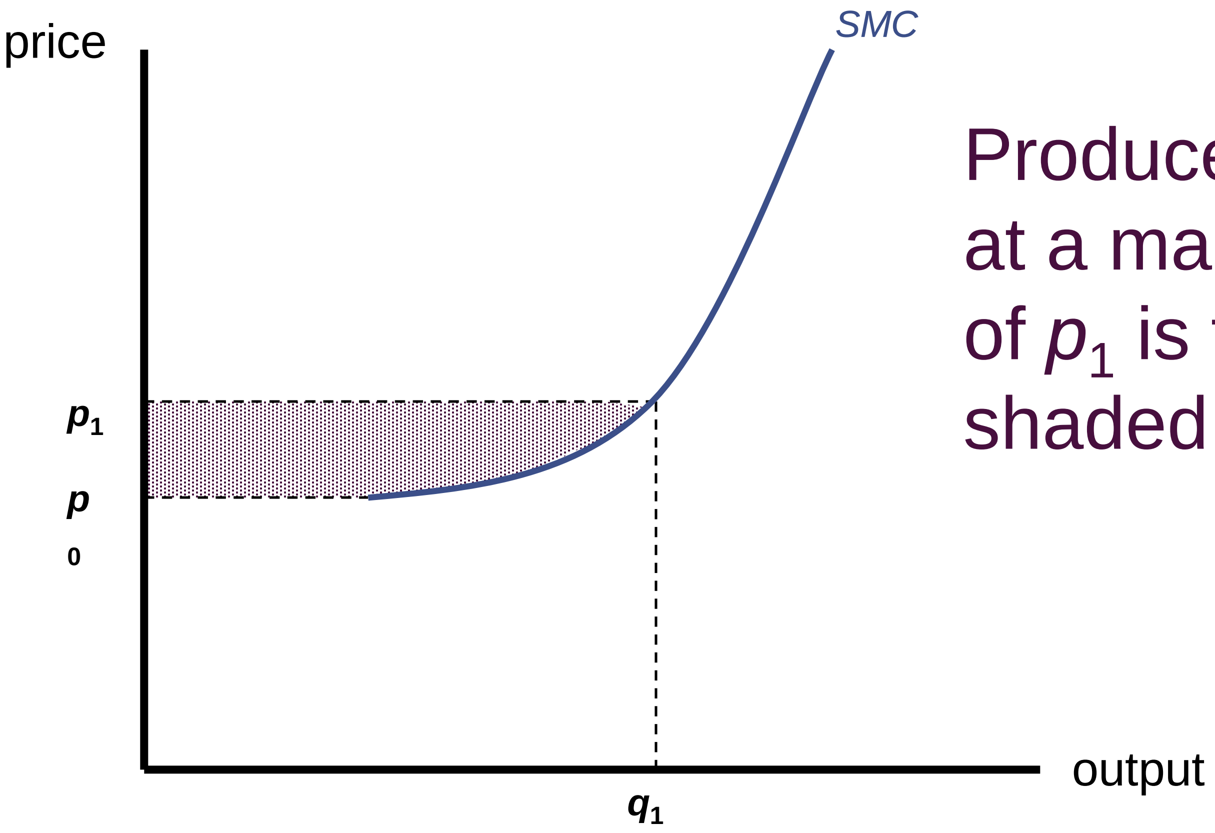
Suppose that the firm's shutdown price is p_0

Producer Surplus in the Short Run

- The extra profits available from facing a price of p_1 are defined to be producer surplus

$$\text{producer surplus} = \Pi(p_1,\ldots) - \Pi(p_0,\ldots) = \int_{p_0}^{p_1} q(p)\,dp$$

Producer Surplus in the Short Run



Producer surplus at a market price of p_1 is the shaded area

Producer Surplus in the Short Run

- Producer surplus is the extra return that producers make by making transactions at the market price over and above what they would earn if nothing was produced
  - the area below the market price and above the supply curve

Producer Surplus in the Short Run

- Because the firm produces no output at the shutdown price, $\Pi(p_0,\ldots) = -vk_1$
  - profits at the shutdown price are equal to the firm's fixed costs
- This implies that

$$\text{producer surplus} = \Pi(p_1,\ldots) - \Pi(p_0,\ldots)$$
$$= \Pi(p_1,\ldots) - (-vk_1) = \Pi(p_1,\ldots) + vk_1$$

  - producer surplus is equal to current profits plus short-run fixed costs

Profit Maximization and Input Demand

- A firm's output is determined by the amount of inputs it chooses to employ
  - the relationship between inputs and outputs is summarized by the production function
- A firm's economic profit can also be expressed as a function of inputs

$$\pi(k,l) = pq - C(q) = pf(k,l) - (vk + wl)$$

Profit Maximization and Input Demand

- The first-order conditions for a maximum are

$$\partial\pi/\partial k = p[\partial f/\partial k] - v = 0$$

$$\partial\pi/\partial l = p[\partial f/\partial l] - w = 0$$

- A profit-maximizing firm should hire any input up to the point at which its marginal contribution to revenues is equal to the marginal cost of hiring the input

Profit Maximization and Input Demand

- These first-order conditions for profit maximization also imply cost minimization
  - they imply that $RTS = w/v$

Profit Maximization and Input Demand

- To ensure a true maximum, second-order conditions require that

$$\pi_{kk} = f_{kk} < 0$$

$$\pi_{ll} = f_{ll} < 0$$

$$\pi_{kk}\,\pi_{ll} - \pi_{kl}^2 = f_{kk}f_{ll} - f_{kl}^2 > 0$$

  - capital and labor must exhibit sufficiently diminishing marginal productivities so that marginal costs rise as output expands

Input Demand Functions

- In principle, the first-order conditions can be solved to yield input demand functions

$$\text{Capital Demand} = k(p,v,w)$$

$$\text{Labor Demand} = l(p,v,w)$$

- These demand functions are unconditional
  - they implicitly allow the firm to adjust its output to changing prices

Single-Input Case

- We expect $\partial l/\partial w \leq 0$
  - diminishing marginal productivity of labor
- The first order condition for profit maximization was

$$\partial\pi/\partial l = p[\partial f/\partial l] - w = 0$$

- Taking the total differential, we get

$$dw = p \cdot \frac{\partial f_l}{\partial l} \cdot \frac{\partial l}{\partial w} \cdot dw$$

Single-Input Case

- This reduces to

$$1 = p \cdot f_{ll} \cdot \frac{\partial l}{\partial w}$$

- Solving further, we get

$$\frac{\partial l}{\partial w} = \frac{1}{p \cdot f_{ll}}$$

- Since $f_{ll} \leq 0$, $\partial l/\partial w \leq 0$

Two-Input Case

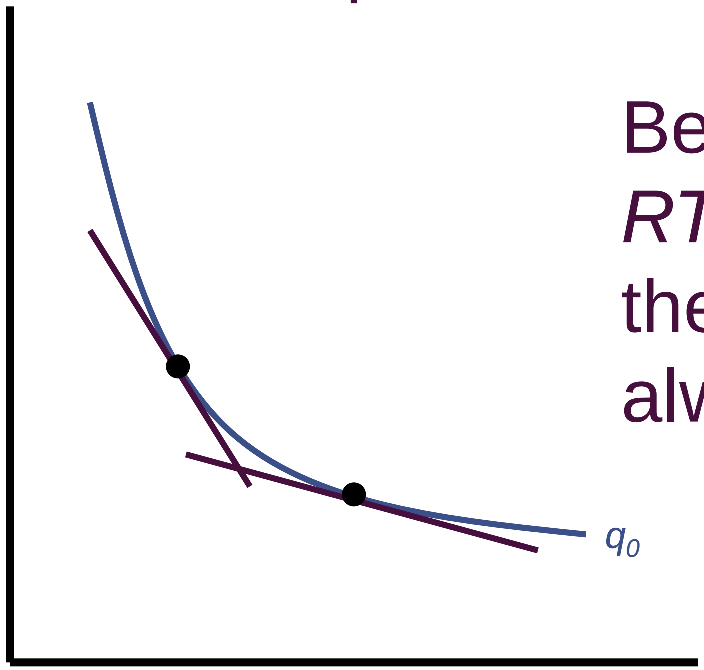
- For the case of two (or more inputs), the story is more complex
  - if there is a decrease in w, there will not only be a change in l but also a change in k as a new cost-minimizing combination of inputs is chosen
    - when k changes, the entire f_l function changes
- But, even in this case, $\partial l/\partial w \leq 0$

Two-Input Case (skipped)

- When *w* falls, two effects occur
  - substitution effect
    - if output is held constant, there will be a tendency for the firm to want to substitute *l* for *k* in the production process
  - output effect
    - a change in *w* will shift the firm's expansion path
    - the firm's cost curves will shift and a different output level will be chosen

Substitution Effect

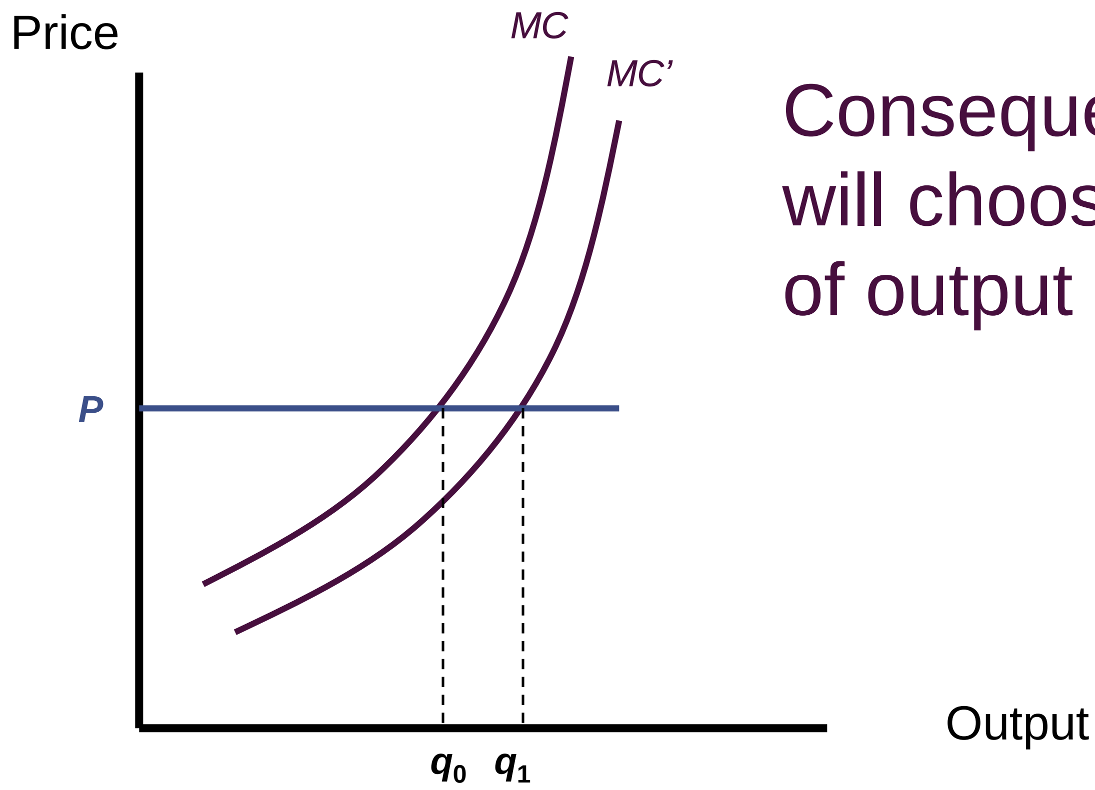
If output is held constant at q_0 and *w* falls, the firm will substitute *l* for *k* in the production process



Because of diminishing *RTS* along an isoquant, the substitution effect will always be negative

Output Effect

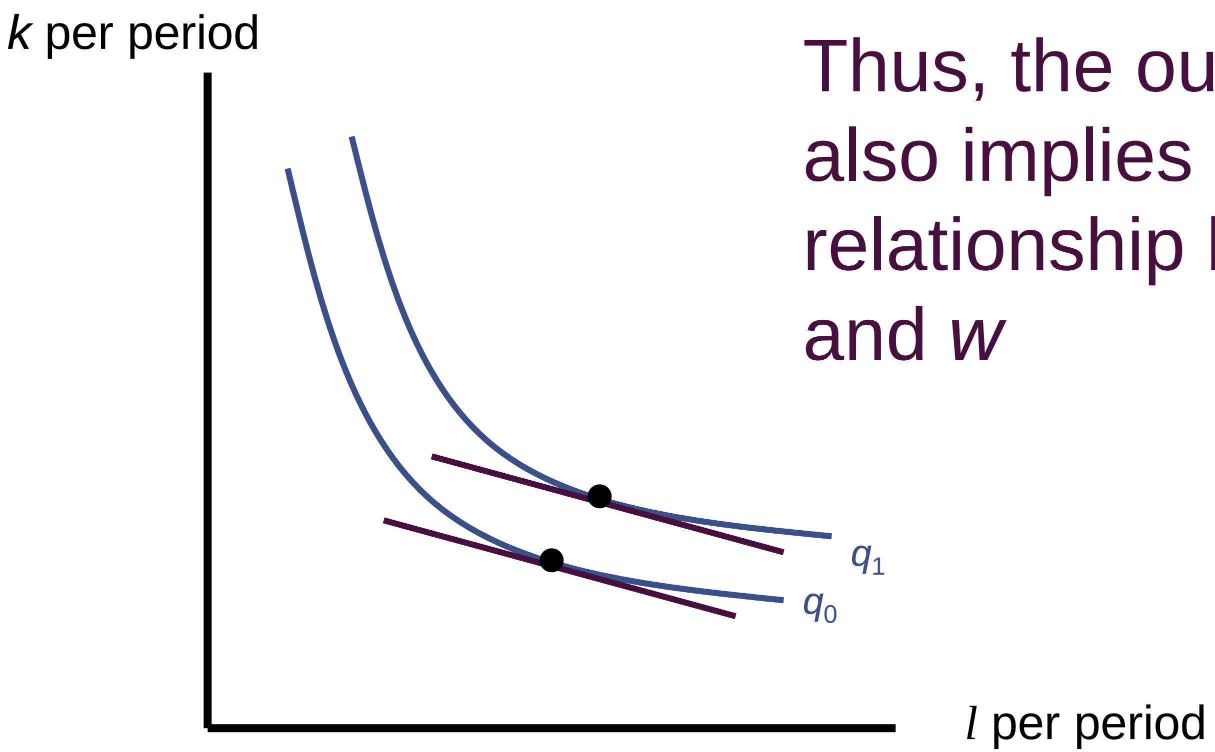
A decline in *w* will lower the firm's *MC*



Consequently, the firm will choose a new level of output that is higher

Output Effect

Output will rise to q_1



Thus, the output effect also implies a negative relationship between *l* and *w*

Cross-Price Effects

- No definite statement can be made about how capital usage responds to a wage change
  - a fall in the wage will lead the firm to substitute away from capital
  - the output effect will cause more capital to be demanded as the firm expands production

Substitution and Output Effects

- We have two concepts of demand for any input
  - the conditional demand for labor, $l^c(v,w,q)$
  - the unconditional demand for labor, $l(p,v,w)$
- At the profit-maximizing level of output

$$l^c(v,w,q) = l(p,v,w)$$

Substitution and Output Effects

- Differentiation with respect to *w* yields

$$\frac{\partial l(p,v,w)}{\partial w} = \underbrace{\frac{\partial l^c(v,w,q)}{\partial w}}_{\text{substitution effect}} + \underbrace{\frac{\partial l^c(v,w,q)}{\partial q} \cdot \frac{\partial q}{\partial w}}_{\text{output effect}}$$

substitution effect + output effect = total effect

Important Points to Note:

- In order to maximize profits, the firm should choose to produce that output level for which the marginal revenue is equal to the marginal cost

Important Points to Note:

- If a firm is a price taker, its output decisions do not affect the price of its output
  - marginal revenue is equal to price
- If the firm faces a downward-sloping demand for its output, marginal revenue will be less than price

Important Points to Note:

- Marginal revenue and the price elasticity of demand are related by the formula

$$MR = p\left(1 + \frac{1}{e_{q,p}}\right)$$

Important Points to Note:

- The supply curve for a price-taking, profit-maximizing firm is given by the positively sloped portion of its marginal cost curve above the point of minimum average variable cost (*AVC*)
  - if price falls below minimum *AVC*, the firm’s profit-maximizing choice is to shut down and produce nothing

Important Points to Note:

- The firm's reactions to the various prices it faces can be judged through use of its profit function
  - shows maximum profits for the firm given the price of its output, the prices of its inputs, and the production technology

Important Points to Note:

- The firm's profit function yields particularly useful envelope results
  - differentiation with respect to market price yields the supply function
  - differentiation with respect to any input price yields the (inverse of) the demand function for that input

Important Points to Note:

- Short-run changes in market price result in changes in the firm's short-run profitability
  - these can be measured graphically by changes in the size of producer surplus
  - the profit function can also be used to calculate changes in producer surplus

Important Points to Note:

- Profit maximization provides a theory of the firm's derived demand for inputs
  - the firm will hire any input up to the point at which the value of its marginal product is just equal to its per-unit market price
  - increases in the price of an input will induce substitution and output effects that cause the firm to reduce hiring of that input