# Games and Contracts
# Lecture 1

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# Introduction

- Game theory studies **how the agents (players) strategically interact with each other.**
  Root: each player's payoff depends on not only his own action but also on others' actions.
- It is a crucial methodology of analyzing human behaviors.
- It has become the core of microeconomic theory.
- Contract theory is heavily based on game theory. So one has to master basic game theory knowledge before thoroughly studying contract theory.

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- The principal(s) and the agent(s) will thus play a **contracting game**.

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- **Asymmetric information**: The agent(s) normally has some hidden actions or information (types). They can take advantage of such asymmetric information for self-benefit. *Eg. insurance or loafing worker.* The principal(s) need to deal with information asymmetry in contracting.
- **Incentive compatibility:** When the principal(s) contracts with the agent(s), she must take into account the interests pursuit of the agent(s). Otherwise, the contracting game cannot reach a sound equilibrium for both or even just the principal.

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- **Bargaining power**: The bargaining power for one party is the power to determine how much his or her objective will weigh in the final objective of principal-agent problem. (Generally, contracting is a bargaining problem. cooperative or coalitional games)
  $\longrightarrow$Normally we assume the principal has *full bargaining power*, so the final objective of principal-agent problem is just the principal's objective. Contracting becomes a noncooperative games. Bargaining power sometimes can determine the role of P and A in difference contexts. The party with full bargaining power will propose contract.
  *Eg. supplier vs retailer.

# Broad Scopes for Application of Contract Theory

- **Employment or Executive Compensation**
- **Auction**
- **Regulation**
- **Nonlinear Pricing**
- **Public good provision**
- **Supply Chain**
- **International relations**
- **Insurance**
- **Voting**

# Math review

- Functions
- Metric Space
- Vector Space and convex sets
- Ordered set
- Euclidean Space
- Differential and Integration
- Probability and Random Variable
- Distribution and Density
- Expectation
- Notable Properties of functions under different structures
- Set-valued functions (maybe)
- Mathematical program and KKT conditions.

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- A function $f$ with *domain* $X$ and *codomain* $Y$ is commonly denoted by

  $$f : X \to Y.$$

  A multi-function $g$ with domain $X$ and codomain $Y$ is commonly denoted by

  $$g : X \twoheadrightarrow Y.$$

# Functions

- The elements of $X$ are called *arguments of* $f$. For each argument $x$, the corresponding **unique** $y$ in the codomain is called *the function value* at $x$ or *the image* of $x$ under $f$. It is written as $f(x)$. But in some cases, people may also use $f(x)$ to denote a function.

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- Intuition: a function is a relation between a set of inputs and a set of permissible outputs with the property that each input is related to exactly one output.

# Metric Space

- A *metric* $d$ on a set $X$ is a function $d : X \times X \to \mathbb{R}$ such that for all $x, y \in X$:
  (1) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$;
  (2) $d(x, y) = d(y, x)$ (symmetry);
  (3) $d(x, y) \leq d(x, z) + d(z, x)$ (triangle inequality).

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- A *metric space* $(X, d)$ is a set $X$ with a metric $d$ defined on $X$.
- Intuition: a metric space is a set where a notion of distance (called a metric) between elements of the set is defined. It contains the *metric (topological) structure* concerning measuring how close two elements of the set will be.

# Vector Space

A *vector space* over $\mathbb{R}$ is a set $V$ together with the operations of addition $V \times V \to V$ and scalar multiplication $\mathbb{R} \times V \to V$ satisfying the following properties:
(1) *Commutativity*: $u + v = v + u$ for all $u, v \in V$ ;
(2) *Associativity*: $(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$ for all $u, v, w \in V$ and $a, b \in \mathbb{R}$;
(3) *Additive identity*: There exists an element $0 \in V$ such that $0 + v = v$ for all $v \in V$;
(4) *Additive inverse*: For every $v \in V$, there exists an element $w \in V$ such that $v + w = 0$;
(5) *Multiplicative identity*: $1v = v$ for all $v \in V$ ;
(6) *Distributivity*: $a(u + v) = au + av$ and $(a + b)u = au + bu$ for all $u, v \in V$ and $a, b \in \mathbb{R}$.

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- addition: some element can be accumulated.
  scalar multiplication: some element can be enlarged by times.
  e.g. monetary reward

# Normed Vector Space

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- A *normed vector space* $(X, \|\cdot\|)$ is a vector space $X$ together with a function $\|\cdot\| : X \to \mathbb{R}$, called a norm on $X$, such that for all $x, y \in X$ and $k \in \mathbb{R}$:
  (1) $0 \leq \|x\| < \infty$ and $\|x\| = 0$ if and only if $x = 0$;
  (2) $\|kx\| = |k| \, \|x\|$;
  (3) $\|x + y\| \leq \|x\| + \|y\|$.

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- Convex set is important for analysis of optimization.

# Ordered set

- A binary relation " $\succsim$ " on a nonempty set $X$ is said to be a *partial order* if
  (1) $x \succsim x$ for every $x \in X$ (reflexive)
  (2) $x \succsim y \succsim x$ implies $x = y$ for every $x, y \in X$ (antisymmetric)
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- Intuition: an ordered set contains the *order structure* between elements of the set.

# Euclidean Space: a well-behaved space

- $\mathbb{R}^n$ is a metric space!
  In Cartesian coordinates, if $p = (p_1, ..., p_n)$ and $q = (q_1, ..., q_n)$ are two points in Euclidean n-space, then the *Euclidean distance* from $p$ to $q$, or from $q$ to $p$ is given by

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- $\mathbb{R}^n$ has a partial order "$\geq$"!
  "$\geq$" will become linear order in $\mathbb{R}$.