Games and Contracts
Lecture 10
Multi-agency Adverse Selection Model

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- The feasible joint contract is $k = (k_i)_{i \in \mathcal{N}} \in \mathcal{K} \subseteq \prod_{i=1}^{n} \mathcal{K}_i$. A given $\mathcal{K}$ may contain some realistic constraints on the contract profiles PL can offer to the agents. Let $k(\omega) = (k_i(\omega))_{i \in \mathcal{N}}$ and $k_{-i}(\omega) = (k_j(\omega))_{j \in \mathcal{N} \setminus \{i\}}$.

The Model: Contracts

Examples

Finite contract sets: There are only finitely many contracts in each $\mathcal{K}_i$. $\mathcal{K}$ will be a compact metric space as any subset of $\prod_{i=1}^{n} \mathcal{K}_i$.

Contract sets for single object: Each bidder i is offered a pair (x_i, p_i). x_i is i's payment to the seller. p_i is the probability that i gets the object. So the joint feasible contract set can be

$$\begin{aligned} \mathcal{K} &= \{(x_1, \cdots, x_n, p_1, \cdots, p_n) \in \mathbb{R}^n \times \mathbb{R}^n : \\ 0 &\leq \sum_i p_i \leq 1, 0 \leq x_i \leq I_i, \text{ for each } i\}, \end{aligned}$$

where $I_i > 0$ is the wealth of bidder i. Obviously, $\mathcal{K}$ is a compact metric space.

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- PL's ex post payoff function over contract and type profiles is $u : \mathcal{K} \times \Theta \to \mathbb{R}$.
- Assume that v_i is continuous on $\mathcal{K} \times \Theta$, and u is continuous on $\mathcal{K}$ and Borel-measurable (continuous) on Θ.

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- Examples:
  VC (knows a project's market value) and innovation firm (knows a project's technology condition)
  Different regions contract on pollution control and each region will enjoy a cleaner environment from others' contract;
  Firm owner's objective (e.g. profit) is based on all the employees' abilities and compensation.

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- $\mu_i(\cdot|\theta_i)$ is the conditional probability on θ_i over θ_{-i}: agent i's interim belief over the other players' private types after learning her own type θ_i.
- For each i and each closed subset E of θ_{-i}, $\mu_i(E|\cdot)$ is continuous on θ_i.
- A$_i$'s interim payoff function $V_i : \mathcal{K} \times \Theta_i \to \mathbb{R}$ is defined by

$$V_i(k, \theta_i) = \int_{\Theta_{-i}} v_i(k, \theta) \mu_{-i}(d\theta_{-i}|\theta_i).$$

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  Let $r = (r_i)_{i \in \mathcal{N}} \in R = \prod_{i=1}^{n} R_i$ and

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- A **(joint-based) general mechanism** is functions

$$\begin{aligned} \mathbf{k} &\in \mathcal{F}(R, \mathcal{K}), \\ \text{where } \mathbf{k} &= (\mathbf{k}_1, \cdots \mathbf{k}_n) : R \to \mathcal{K} \text{ satisfying } (\mathbf{k}_1(r), \cdots \mathbf{k}_n(r)) \in \mathcal{K} \end{aligned}$$

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- A **report function** for agent i is a function $\rho_i : \theta_i \to R_i$ specifying agent i's report given each type of i.
- If $R_i = \theta_i$ for each i when the rest of the setting remains unchanged, such mechanisms are called **direct mechanisms**.

Contracting game over mechanisms

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- At Stage 3, through the pre-offered mechanism, the principal assigns contracts to the agents according to their reports.
- At Stage 4, outcomes are realized and the contracts are implemented.

- The agents' participation strategy profile ρ is said to be an **ex post equilibrium (EPE)** under a joint-based general mechanism $\mathbf{k}$ if for each $i \in \mathcal{N}$ and each $\theta \in \Theta$,

$$v_i(\mathbf{k}(\rho(\theta)), \theta) \geq v_i(\mathbf{k}(\rho_i^{'}(\theta_i), \rho_{-i}(\theta_{-i})), \theta),$$

for all ρ_i'. Moreover, such $\mathbf{k}$ is referred to as a (joint-based) **ex post general Mechanism**.

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- A (joint-based) direct mechanism $\mathbf{k}$ is **ex post incentive compatible (EPIC)** if it induces truthful reporting as the EPE for all the agents, i.e. for each $i \in \mathcal{N}$ and each $\theta \in \Theta$,

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- If the agents have private valuations, EPE or EPIC will reduce to DSE or DSIC.

- The agents' participation strategy profile $\rho \in \Gamma$ is said to be a **Bayesian Nash equilibrium (BNE)** under a joint-based general mechanism $\mathbf{k}$ if for each $i \in \mathcal{N}$ and each $\theta_i \in \Theta_i$,

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- **(P1)** joint-based ex post general mechanism design problem

$$\max_{\mathbf{k}\in\mathcal{F}(R,\mathcal{K})} \max_{\rho\in\Gamma} \int_{\Theta} u(\mathbf{k}(\rho(\theta)),\theta),\theta)\mu(d\theta)$$

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- **(P4)** joint-based BIC mechanism design problem:

$$\max_{\mathbf{k}\in\mathcal{F}(\Theta,\mathcal{K})} \int_{\Theta} u(\mathbf{k}(\theta),\theta)\mu(d\theta)$$

s.t. $\mathbf{k}$ is *BIC*.

General Revelation Principle

- **Ex Post Implementation Version**
  Given any EPE ρ of the subgame played by the agents under any joint-based ex post general mechanism $\mathbf{k} \in \mathcal{F}(R, \mathcal{K})$ in **(P1)**, there exists a joint-based EPIC mechanism $\widetilde{\mathbf{k}} \in \mathcal{F}(T, \mathcal{K})$ in **(P2)** in which the principal obtains the same expected payoff as in the EPE ρ of the given ex post general mechanism $\mathbf{k}$. Furthermore, the optimal joint-based EPIC mechanism solving **(P2)** is also optimal in the class of all joint-based ex post general mechanisms.

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- **Bayesian Implementation Version**
  Given any BNE ρ of the subgame played by the agents under any joint-based ex post general mechanism $\mathbf{k} \in \mathcal{F}(R, \mathcal{K})$ in **(P3)**, there exists a joint-based BIC mechanism $\widetilde{\mathbf{k}} \in \mathcal{F}(T, \mathcal{K})$ in **(P4)** in which the principal obtains the same expected payoff as in the BNE ρ of the given ex post general mechanism $\mathbf{k}$. Furthermore, the optimal joint-based BIC mechanism solving **(P3)** is also optimal in the class of all joint-based ex post general mechanisms.

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- **Intuition**: One can restrict out attention to IC direct mechanisms out of general mechanism design with no loss of generality.
- Myerson (1982 JME) even gives a very general version of revelation principle with both adverse selection and moral hazard!

Applications: Auction (Myerson 1981)

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- There are n potential bidders for the object. The participants' expected utilities are

$$V_i = x_i \theta_i - t_i.$$

x_i - participant i's probability of receiving the good in the auction
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- Assume reservation utility of each bidder is $r_i \equiv 0$.

Applications: Auction (Myerson 1981)

- The principal's direct mechanism is given by a pair of functions $\mathbf{x} : \Theta \to \mathcal{X}$ and $\mathbf{t} : \Theta \to \mathbb{R}^n$ where $\mathcal{X} = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_i \geq 0 \text{ and } \sum_{i=1}^{n} x_i \leq 1\}$

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- Characterization: An auction mechanism is BIC and IR (under standard definitions) iff
  $\int_{\Theta_{-i}} \mathbf{x}(h, \theta_{-i}) dF_{-i}(\theta_{-i})$ is nondecreasing in θ_i,
  $\widehat{V}_i(b_i; \mathbf{x}, \mathbf{t}_i) - \widehat{V}_i(a_i; \mathbf{x}, \mathbf{t}_i) =$
  $\int_{a_i}^{b_i} \int_{\Theta_{-i}} \mathbf{x}(\theta) dF_{-i}(\theta_{-i}) d\theta_i, \forall a_i, b_i \in \Theta_i$,
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  $\int_{a_i}^{b_i} \int_{\Theta_{-i}} \mathbf{x}(\theta) dF_{-i}(\theta_{-i}) d\theta_i, \forall a_i, b_i \in \Theta_i,$
  $\widehat{V}_i(\underline{\theta}_i; \mathbf{x}, \mathbf{t}_i) \geq r_i.$
- In optimal auction design, PL's objective function is to maximize expected payments considering the principal's own value of the object, which we take to be θ_0. Specifically,
  $\max_{\mathbf{x},\mathbf{t}} \int_{\Theta} (1 - \sum_{i}^{n} \mathbf{x}_i(\theta))\theta_0 + \sum_{i}^{n} \mathbf{t}_i(\theta) dF(\theta)$
  s.t. IC and IR.

If

$$\mathbf{x}^* \in \arg\max_{\mathbf{x}} \int_{\Theta} \theta_0 + \sum_{i=1}^{n} (\mathbf{x}_i(\theta)(\theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} - \theta_0) dF(\theta)$$

s.t. $\int_{\Theta_{-i}} \mathbf{x}_i(h, \theta_{-i}) dF_{-i}(\theta_{-i})$ is increasing in h on Θ_i,
$\mathbf{x}(\theta) \in \mathcal{X}, \forall \theta \in \Theta$.
In addition, $\forall i \in \mathcal{N}, \theta \in \Theta$,
$\mathbf{t}_i^*(\theta)$ is such that

$$\int_{\Theta_{-i}} \mathbf{t}_i^*(\theta) dF_{-i}(\theta_{-i}) = \int_{\Theta_{-i}} [\int_{\underline{\theta}_i}^{\theta_i} \mathbf{x}_i^*(s_i, \theta_{-i}) ds_i - \mathbf{x}_i^*(\theta)\theta_i] dF_{-i}(\theta_{-i}).$$

Then such a combination $(\mathbf{x}^*, \mathbf{t}^*)$ is the **optimal auction mechanism.** Moreover, the optimal objective value of PL is always equal to
$\int_{\Theta} \theta_0 + \sum_{i=1}^{n} (\mathbf{x}_i^*(\theta)(\theta_i - \frac{1-F_i(\theta_i)}{f_i(\theta_i)} - \theta_0) dF(\theta)$
In this case, $\mathbf{x}^*$ uniquely determines the optimum.

Let $J_i(\theta_i) \equiv \theta_i - \frac{1-F_i(\theta_i)}{f_i(\theta_i)}$

Assume that each $\frac{1-F_i(\theta_i)}{f_i(\theta_i)}$ is nonincreasing. Then the optimal auction has

chosen such that

$$\mathbf{x}_i^*(\theta) = \{ \begin{array}{lll} 1 & \textit{if} \;\; J_i(\theta_i) > \max\limits_{k \neq i} J_k(\theta_k) \text{ and } J_i(\theta_i) \geq \theta_0 \\ \in [0,1] & \textit{if} \;\; J_i(\theta_i) = \max\limits_{k \neq i} J_k(\theta_k) \text{ and } J_i(\theta_i) \geq \theta_0 \\ 0 & \qquad\qquad \textit{otherwise} \end{array}$$

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- There is no principal.
- Instead think of the optimization problem as the design of a mechanism by the buyer and seller before they know their types, but under the condition that after learning their types, either party may walk away from the agreement.
- Additionally, it is assumed that money can only be transferred from one party to the other. There is no outside party that can break the budget.

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- **Theorem** For any $\mathbf{x}$, there there exists a transfer function $\mathbf{t}$ such that $(\mathbf{x}, \mathbf{t})$ is BIC and IR iff

$$E_{v,c}[\mathbf{x}(c,v)\{(v - \frac{1-F_2(v)}{f_2(v)}) - (c + \frac{F_1(c)}{f_1(c)})\}] \geq 0, \qquad (1)$$

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  subject to monotonicity and (1).
- We will ignore monotonicity and check our solution to see that it is satisfied.

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- Let μ be the constraint (1). Bringing the constraint into the objective function and simplifying, we have

$$\max_{\mathbf{x}} E_{c,v}[\mathbf{x}(c,v)\{(v-c)-\frac{\mu}{1+\mu}(\frac{1-F_2(v)}{f_2(v)}-\frac{F_1(c)}{f_1(c)})\}]$$

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- If we assume that the monotone hazard-rate condition is satisfied for both type distributions, then this is appropriately monotonic and we have a solution to the full program. Note that if $\mu > 0$, there will generally be inefficiencies in trade.