Game Theory and Economics of Contracts
Lecture 2
Mathematics Preparation

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Spring 2015

Notable Properties of functions under metric structure

- Continuity

Given X and Y are both metric spaces. A function $f : X- > Y$ is **continuous** if for any open set (or equivalently, closed set) E in Y, the **pre-image** of E, $f^{-1}(E) = \{x : f(x) \in E\}$ is also open (or equivalently, closed) in X.
In the language of limits, a function f is continuous at point $x_0 \in X$ if $\lim_{x \to x_0} f(x)$ exists and $\lim_{x \to x_0} f(x) = f(x_0)$. If f is continuous at every point $x \in X$, f is said to be continuous.

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- Intuition: In mathematics, a continuous function is a function for which, intuitively, "small" changes in the input result in "small" changes in the output.
- Most payoff (utility) functions are assumed to be continuous.

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- Continuous function can preserve **compactness**.

In a metric space X, a set $A \subseteq X$ is **compact** if every sequence in A has a convergent subsequence in X.
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- A set E in a metric space is **closed** if any convergent sequence in E must converge to a point also in E.
- A set E in a metric space is **bounded** if the supremum of all the distances between two points in E is finite.
  supremum (resp., **infimum**) for a subset E of arbitrary partially ordered sets $(X, \leq)$ is an element u in X such that
  (1) $x \leq u$ (resp., $x \geq u$) for all $x \in X$, and
  (2) for any v in X such that $x \leq v$ (resp., $x \geq v$) for all x in X, $u \leq v$(resp., $u \geq v$).
  Be careful: For a given set, its supremum and infimum always exist. But its maximum and minimum may not exist!

Notable Properties of functions under metric structure

- The converse is not always true. But **Heine-Borel Theorem** indicates $\mathbb{R}^n$ is a good one.
  Any set in $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

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  Any set in $\mathbb{R}^n$ is compact if and only if it is closed and bounded.
- Every closed subset of a compact space is compact.
- Given X and Y are both metric spaces. If a function $f : X -> Y$ is a continuous function, then the image of every compact subset E in X under f, $f(E) = \{f(x) \in Y : x \in X\}$, is also compact in Y.

Some important remarks on Math

- In metric spaces, we may use sequences as a very important tool for many proofs in game theory and contract theory.

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- When we use any propositions or theorems in math, you should remember the results as well as under what conditions/hypotheses the results can come up.
- Refer to a note I wrote about Reasoning, Proof and Writing in Mathematics on my website teaching page.

Notable Properties of functions under metric structure

- Semi-Continuous function

In mathematical analysis, semi-continuity is a property of extended real-valued functions that is weaker than continuity.
Given a metric space X, a function $f : X \to [-\infty, \infty]$ is **upper (resp., lower) semi-continuous** if for each $c \in \mathbb{R}$, $\{x \in X : f(x) \leq c\}$ (resp., $\{x \in X : f(x) \geq c\}$) is closed.
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- A function f is lower semicontinuous if and only if (short for iff) $-f$ is upper semicontinuous, and vice versa.
- A real valued function is continuous iff it is both upper and lower semicontinuous.

Notable Properties of functions under metric structure

- More important: (Aliprantis 2006 Theorem 2.43)
  A real-valued lower (resp., upper) semicontinuous function one a compact pace attains a minimum (resp. maximum) value and the nonempty set of minimizers (resp. maximizers) is compact.
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  This is a generalization of **Weierstrass Theorem**.
- Due to uncertainty and interaction, sometimes the payoff function of interest may be semicontinuous.

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Set-valued functions
upper hemicontinuity
lower hemicontinuity
Measurability
see the book "Infinite Dimensional Analysis" (Aliprantis 2006)
Chapter 17 and 18
Or Himmelberg "On measurable relations" and other relevant papers/surveys

Notable Properties of Functions under Linear Algebra Structure

- Concave and Convex Functions
  A real-valued function f on a **convex set** X in a vector space is **concave (resp., convex)** if, for any x and y in X and for any $t \in [0, 1]$, $f(tx + (1 - t)y) \geq$ (resp. $\leq$) $tf(x) + (1 - t)f(y)$.

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- A function is **strictly concave (resp. convex)** if $f(tx + (1-t)y) >$ (resp., $<$) $tf(x) + (1-t)f(y)$, for any $t \in (0,1)$ and $x \neq y$.

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- If X is a Euclidean space and f is concave or convex, f must be continuous. If f is also **of 2C** (has continuous second order derivatives), then one can use Hessian matrix to analyze concavity or convexity. Check your mathematics for economics books!

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- A function that is both convex and concave is **linear**.

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- Quasi-Concave and Quasi-Convex Functions
  A function $f : S \rightarrow \mathbb{R}$ defined on a **convex subset** X of a vector space is **quasiconcave (resp., quasiconvex)** if for any x and y in X and for any $t \in [0, 1]$, $f(tx + (1 - t)y) \geq \min\{f(x), f(y)\}$(resp., $\leq \max\{f(x), f(y)\}$).

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- A function that is both quasiconvex and quasiconcave is **quasilinear**.
- Any monotonic function is both quasiconvex and quasiconcave.

Notable Properties of Functions under Order Structure

Preliminaries
Let $(X, \geq)$ and $(T, \geq)$ be partially ordered sets. We may view X as the set of choice variables and T as the set of parameters.
In any POSET define $x \vee y = \sup\{x, y\}; x \wedge y = \inf\{x, y\}$.

- Cardinal Complementarity Conditions: Supermodularity
  A function $f : X \to \mathbb{R}$ is **(strictly) Supermodular** if
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- If X is a subset of $\mathbb{R}^n$ and f is of 2C, $f : X \rightarrow \mathbb{R}$ is **(strictly) supermodular** iff $\frac{\partial^2 f(x)}{\partial x_i \partial x_j}(>) \geq 0, \forall i \neq j$.
Compare this with convexity!

Notable Properties of Functions under Order Structure

- Cardinal Complementarity Conditions: Increasing Difference
  Consider a function $f : X \times T \rightarrow \mathbb{R}$.
  f has **increasing differences (ID)** in $(x; t)$ if $\forall x' > x, t' > t$

  $$f(x', t') - f(x', t) \geq f(x, t') - f(x, t)$$

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- In other words, the difference $h(x) = f(x, t') - f(x, t)$ is an increasing function.
  If X is a subset of $\mathbb{R}^n$, T is a subset of $\mathbb{R}^m$, and f is of 2C, then f has (strictly) *ID* in $(x; t)$ iff $\frac{\partial^2 f(x,t)}{\partial x_i \partial t_j} (>) \geq 0$, $\forall i = 1, \cdots, n, j = 1, \cdots, m$.

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- Why are they **cardinal conditions**?
  Not invariant under **monotone transformation**. For any strictly increasing transformation g, $g \circ f$ may not necessarily be supermodular or have ID.
- **Intuition:** They represent the concept of "strategic complementarity". Consider f as the objective function, Eg. Utility function or profit function.
  Supermodularity: Any increase in some subset of the choice variables increases the marginal return of having more of the remaining choice variables and therefore make the increase in them desirable.
  ID: Any increase in the parameter increases the marginal return of having more of choice variable and therefore make the increase in it desirable.

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- Ordinal Complementarity Conditions: Quasi-supermodularity
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- A function $f : X \longrightarrow \mathbb{R}$ is **strictly quasi-supermodular** if $f(x) \geq f(x \wedge x')$ implies $f(x \vee x') > f(x')$, $\forall x, x' \in X$.

Ordinal Complementarity Conditions: Single crossing properties

- Consider a function $f : X \times T \to \mathbb{R}$.
  f satisfies the **single crossing property (SCF)** in $(x; t)$ if $\forall x' > x, t' > t$

$$f(x', t) \geq (>) f(x, t) \implies f(x', t') \geq (>) f(x, t')$$

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- Supermodularity implies Quasi-supermodularity. ID implies SCF.

- Why are they **ordinal conditions**?
  Invariant under **monotone transformation**. For any strictly increasing transformation g, $g \circ f$ must still be quasi-supermodular or have SCF.
- **Intuition:** They represent the weak kinds of "strategic complementarity".
  Quasi-supermodularity: if an increase in some subset of the choice variables is desirable at some level of the remaining choice variables, it will remain desirable as the remaining variables also increase.
  SCP: A given increase in the decision variable is desirable when the parameter is low, the same increase will continue to be desirable when the parameter is high.

For more advanced content, please refer to

- Supermodularity and Complementarity in Economics: An Elementary Survey (R. Amir 2005)
- Monotone Comparative Statics, (P. Milgrom and C. Shannon 1994)
- Supermodularity and complementarity, (M.D. Topkis 1998)

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- In the environment with uncertainty, we want to compare riskiness between two random objects (normally, we consider random variables)
- One simple way is to compare the variances of two random variables. But it is not sufficient in decision theory. It ignores the decision makers' preferences (utilities).
- A better tool is *Stochastic Dominance*, which is a kind of stochastic order and it contains preference consideration.

- For any random variable X and X' with cumulative probability functions F and G, X or F **first-order stochastically dominates (or is less risky than)** X' or G iff $F(x) \leq G(x)$, $\forall x$.

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- For any distributions F and G, F first-order stochastically dominates G iff the decision maker weakly prefers F to G under every weakly increasing utility function u, i.e., $\int u(x)dF \geq \int u(x)dG$.

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- For any distributions F and G, F **second-order stochastically dominates (or is less risky than)** G iff the (risk-averse) decision maker weakly prefers F to G under every every weakly increasing **concave** utility function u, i.e., $\int u(x)dF \geq \int u(x)dG$.

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- Refer to Stochastic Dominance and Expected Utility: Survey and Analysis (Haim Levy 1992)

Karush–Kuhn–Tucker conditions

Consider the following nonlinear optimization problem:
$\max_{x=(x_{1,} \cdots ,x_n)} f(x_{1,} \cdots , x_n)$
s.t. $g^j(x_{1,} \cdots , x_n) \leq C_j$, $j = 1, \cdots , m$,
where $x_i \in \mathbb{R}$ is the optimization variable, f is the real-valued objective function, each real-valued function $g^j(x_{1,} \cdots , x_n)$ is the inequality constraint. $C_j \in \mathbb{R}$.
Let $\mathcal{L} = f(x_{1,} \cdots , x_n) + \sum_j \lambda_j (C_j - g^j(x_{1,} \cdots , x_n))$. Let $\lambda = (\lambda_{1,} \cdots , \lambda_m)$.

- Suppose that the objective function f and the real-valued constraint functions g^j are continuously differentiable at a point x^*. **KKT conditions** are necessary conditions for that x^* is a maximizer: There exists λ^* such that

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_i}(x^*, \lambda^*) &= 0, \forall i \quad \text{(Stationarity)} \\
\frac{\partial \mathcal{L}}{\partial \lambda_j}(x^*, \lambda^*) &\geq 0, \forall j \quad \text{(Primal feasibility)} \\
\lambda_j^* &\geq 0, \forall j \quad \text{(Dual feasibility)} \\
\lambda_j^* &= 0, \text{if} \frac{\partial \mathcal{L}}{\partial \lambda_j}(x^*, \lambda^*) > 0, \\
\lambda_j^* &> 0, \text{if} \frac{\partial \mathcal{L}}{\partial \lambda_j}(x^*, \lambda^*) = 0, \forall j \\
& \quad \text{(Complementary slackness)}
\end{aligned}
$$

- Sufficient conditions for that x^* is a maximizer is $\mathcal{L}$ is concave in x.