Game Theory and Economics of Contracts
Lecture 3
Basics in Game Theory (1)

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Basic Elements of Game Theory

- Game theory has a very general scope for all of social sciences. It can serve as the methodology and offer insights into any economic, political, or social situation that involves individuals who have different goals or preferences.
- A **game** is a formal representation of a social situation in which a number of individuals interact in a setting of *strategic interdependence*.
- **Strategic interdependence**: each individual's welfare depends not only on her own actions but also on the actions of the other individuals. Moreover, the actions that are best for her to take may depend on what she expects the other players to do.

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3. The **outcomes:** For each possible set of actions by the players, what is the outcome of the game?
4. The **payoffs**: What are the players' preferences (i.e., utility functions) *over the possible outcomes*?

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3. Outcomes: If we meet each other, we get to enjoy each other's company at lunch. Otherwise, we must eat alone.
4. Payoffs: We each attach a monetary value of 100RMB to the other's company. Our payoffs are each 100 RMB if they meet, 0 if we do not.
   Each players optimal action depends on what he thinks the other will do!

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- Noncooperative games: in which the sets of possible actions of individual players are primitives.
- Cooperative games: in which the sets of possible *joint* actions of *groups of* players are primitives.
- We will mainly deal with noncooperative games.

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- Extensive Form Representation of a Game
  It is good for Sequential games:
  The games specifying the possible orders of events, in which each player can consider his plan of action not only at the beginning of the game but also whenever he has to make a decision.

Strategic Form Games

- An n player **strategic form game** is specified by an m-tuple of pairs, $G = (X_i, u_i)_{i=1}^{n}$, where X_i is the set of (pure) strategies, $x_i^{\prime}s$,(equivalent to actions in simultaneous-move games) available to player $i \in I = \{1, 2, ..., n\}$ and $u_i : \prod_{i \in I} X_i \to \mathbb{R}$ is the payoff function for player i.

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- A strategic form game is **zero sum** if $\sum_i u_i(x) = 0, \forall x = (x_1, \cdots, x_n) \in X = \prod_{i \in I} X_i$.

Dominant Strategies

- Given n player strategic form game $G = (X_i, u_i)_{i=1}^n$ strategy $\widehat{x}_i \in X_i$ belonging to player i is **strictly dominant** if

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for all $(x_i, x_{-i}) \in X$ with $\widehat{x}_i \neq x_i$.

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		L	R
player1	U	$(3, 0)$	$(0, -4)$
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- For player 1, U is a strictly dominant strategy - player 1 will play U no matter what player 2 plays. But player 2 has no strictly dominant strategy. Why?

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- Example 2

		player2		
		L	M	R
	U	$(3, 0)$	$(0, -5)$	$(0, -4)$
player1	C	$(1, -1)$	$(3, 3)$	$(-2, 4)$
	D	$(2, 4)$	$(4, 1)$	$(-1, 8)$

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- 1's strategy C is strictly dominated by 1's strategy D. So 1 will never play C. 2's strategy M is strictly dominated by 2's strategy R. So 2 will never play M. Eliminating 1's C and 2's M, then reduces the game in Example 2 to the game in Example 1.

Example 3

		player2	
		L	R
player1	U	$(1,1)$	$(0,0)$
	D	$(0,0)$	$(0,0)$

For player 1, D is weakly dominated by U. For player 2, R is weakly dominated by L.

Dominant Strategies: Iterative Elimination

- Formalize the procedure of **iteratively eliminating strictly dominated strategies** as follows: for each player i let X_i^{sn} denote those strategies of player i surviving after the nth round of elimination. Thus, $x_i \in X_i^{sn}$ if $x_i \in X_i^{sn-1}$ is not strictly dominated in X^{n-1}.

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- $X_i^{wn} \subseteq X_i^{sn}$.

Dominant Strategy Equilibrium

- **Solution concepts:** A solution concept is a formal rule for predicting how a game will be played. These predictions are called "solutions", and describe which strategies will be adopted by players and, therefore, the result of the game. The most commonly used solution concepts are equilibrium concepts,

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- Dominant strategy equilibrium is a very strong/robust solution concept.
  Given n player strategic form game $G = (X_i, u_i)_{i=1}^n$, joint strategy (or strategy profile) $\widehat{x} \in X$ is a (pure strategy) **dominant strategy equilibrium (DSE)** of G if for each player i,
  $$u_i(\widehat{x}_i, x_{-i}) \geq u_i(x_i, x_{-i}), \forall x \in X.$$

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		*player*2	
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*player*1	U	$(3,0)$	$(0,-4)$
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Its DSE?

Nash Equilibrium

- Given n player strategic form game $G = (X_i, u_i)_{i=1}^n$, joint strategy (or strategy profile) $\widehat{x} \in X$ is a (pure strategy) **Nash Equilibrium (NE)** of G if for each player i,

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- Intuition: nobody will have incentive to deviate from NE strategies. NE represents a "double coincidence" between the player's optimal choices. In NE, one player's NE strategy is mutually the best response to the other players' NE strategy.

How to compute NE?

- In finite cases, theoretically, one can start with supposing one player's optimal choice (equilibrium strategy). Then find others' responding optimal choices (equilibrium strategy). Next, under such responding optimal choices, check whether the initial one's supposed optimal choice is indeed the optimal choice. If it is, then one can identify such profile to be an NE. One may have to check all the strategy profiles one by one.

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- In infinite cases, there could be a system of equations to corresponding every player's best response to other players' strategies. Solution to the system of equations will yield the NE.

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- Two members of a criminal gang are arrested and imprisoned. Each prisoner is in solitary confinement with no communication with the other (simultaneous-move game). The police admit they do not have enough evidence to convict the pair on the principal charge. They design a plan of years served for the prisoners.

- That yields a payoff matrix for the prisoners

		B	
		silent	*confess*
A	*silent*	$(-2, -2)$	$(-10, -1)$
	confess	$(-1, -10)$	$(-5, -5)$

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- From another angle, this outcome is best for the police if we introduce them as a player as well! The power of economic design. Fudan Case!

Example 6: Coordination Game or War of Sex

- Two people wish to hang out together in a weekend, and they will agree on the more desirable activity (Mozart concert or Basketball game). A game that captures this situation is given below.

		Betty	
		Mozart	*Basketball*
Adam	*Mozart*	$(2, 2)$	$(0, 0)$
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- The game has two Nash equilibria: (Mozart, Mozart) and (Basketball, Basketball). The players have a mutual interest in reaching one of these equilibria, namely (Mozart, Mozart); however, the notion of Nash equilibrium does not rule out a steady state in which the outcome is the inferior equilibrium (Basketball, Basketball).

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- Think about this game when you are dating someone! =)

Example 7: Cournot Competition

- Cournot Competition is a canonical example of *infinite games* with NE
  There are two Cournot Duopoly firms: firm 1 and 2. The two firms produce identical products with firm 1 producing an amount of q_1 units and firm 2 producing an amount of q_2 units. The total production by both firms will be denoted by q, i.e., $q = q_1 + q_2$. Let $p(q) = A - q$ be the price per unit of the product in the market, where A is a fixed number. Assume that the total cost to firm i of producing the output q_i is $c_i q_i$, where the c_i are positive constants.

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- The strategy set of each player is the set of positive quantities that a firm can choose. That is, the strategy set of each player is $[0, \infty)$. The payoff function of firm i is simply its profit function $\pi_i(q_1, q_2) = (A - q_1 - q_2)q_i - c_i q_i$.

- The problem faced by the firms is how to determine how much each one of them should produce in order to maximize profit—notice that the profit of each firm depends on the output of the other firm. It must satisfy

$$\frac{\partial \pi_i(q_1, q_2)}{\partial q_i} = 0$$

This yields i's best response q_i to q_{-i}.

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- According to the NE Test, the Nash equilibrium, (q_1^*, q_2^*) is the solution of the system
- $\frac{\partial \pi_1(q_1,q_2)}{\partial q_1} = -2q_1 - q_2 + A - c_1 = 0$
  $\frac{\partial \pi_2(q_1,q_2)}{\partial q_2} = -q_1 - 2q_2 + A - c_2 = 0$
  $\Longrightarrow$ Best responses: $q_1^* = \frac{A-c_1-q_2^*}{2}, q_2^* = A - c_1 - 2q_2^*$.
  $q_1^* = \frac{A+c_2-2c_1}{3}, q_2^* = \frac{A+c_1-2c_2}{3}$ Cournot (Nash) equilibrium.

DSE vs. NE

- DSE is stronger than NE. DSE is imposed on stricter conditions as a refined solution concept of NE.
- Iteratively eliminating strictly dominated strategies will not eliminate any NE.
- But iteratively eliminating weekly dominated strategies may.

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- Theorem 1 (Debreu 1952; Glicksberg 1952; Fan 1952) Consider a strategic-form game whose strategy spaces $X_i's$ are nonempty compact convex subsets of an Euclidean space (or even normed vector space). If each payoff function u_i is continuous (or even upper semi-continuous) in x and quasi-concave in x_i, then there exists a pure strategy NE.
- This is an existence result concerning infinite games. (Why?) Meanwhile, Any finite game has a mixed strategy equilibrium but not necessarily a PSNE. But most applications deal with pure strategies.
- This existence result requires metric structure and vector (linear algebra) structure but no order structure (related to supermodular games and Tarksi fixed-point theorem).

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- For more advance theories about the more general existence result, refer to MR Baye, G Tian, J Zhou (1993 RES), PJ Reny (1999 Econometrica), etc.

Games in Strategic Form with Incomplete Information

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- The presence of incomplete information raises the possibility that we may need to consider a player's beliefs about other players' preferences, her beliefs about their beliefs about her preferences, and so on.
- So standard approach to study games with incomplete information will introduce (payoff) types. Each player's type can be viewed as a random variable (parameter) in her (maybe also others') payoff function(s). Its realization (real type) is only observable to the player herself.

- In addition to strategic form with complete information, we now also assume each player can be one of a finite number of unobservable types $t_i \in T_i$, where T_i is player $i's$ type set. All that is known about the profile of types $t = (t_1, ..., t_n) \in T_1 \times \cdots \times T_n = T$ prevailing in the game is that type profile $t \in T$ will prevail with probability $P(t) \in [0, 1]$. P is called the prior (beliefs).

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- Accordingly, each player $i's$ strategy becomes a function $x_i : T_i \to A_i$, where A_i denotes the set of (deterministic) actions available to i. Player $i's$ strategy specifies what action i will take after observing her own real type. Write $A = \prod_{i=1}^{n} A_i$, $x(t) = \prod_{i=1}^{n} x_{-i}(t_{-i})$ and $x_{-i}(t_{-i}) = \prod_{j \neq i}^{n} x_j(t_j)$.

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- An n player strategic form game with incomplete information is specified by an m-tuple of 4-tuples, $H = (P, T_i, X_i, u_i)_{i=1}^{n}$ where T_i is a set of i-th player types, X_i is the i-th player's strategy set, $u_i : A \times T \to \mathbb{R}$ is the ith player's payoff function defined over strategy and type profiles, and p is a probability measure (distribution) defined on T such that $P(t) > 0$ for all $t \in T$.

Games in Strategic Form with Incomplete Information

- If one player's type is only an argument in her own payoff function, then such situations are called **Private Valuation (value)** situations; If one player's type is an argument not only in her own payoff function but also in other players' payoff functions, then such situations are called **Interdependent Valuation (value)** situations.

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- Here u_i is a general version permitting interdependence valuation. It will reduce to interdependence valuation if u_i is the function from X and only T_i to $\mathbb{R}$.
- If the information structure of the game is sufficiently rich, we may allow Bayesian update beliefs to be the common knowledge – since each player knows his own real type t_i, player i can compute a conditional probability $P(\cdot|t_i)$: $T_{-i} \longrightarrow [0, 1]$ for the types of the other players. $P(\cdot|t_i)$ is normally assumed to be derived from the prior P.

Bayesian Nash Equilibrium

- Given an n player strategic form game with incomplete information $H = (P, T_i, X_i, u_i)_{i=1}^n$. Suppose that Bayesian update beliefs are available as common knowledge. A strategy profile $\widehat{x}$ is (pure strategy) **Bayesian Nash Equilibrium (BNE)** if

$$\begin{aligned} & \int_{T_{-i}} u_i(\widehat{x}_i(t_i), \widehat{x}_{-i}(t_i), t) dP(t_{-i}|t_i) \\ \geq \quad & \int_{T_{-i}} u_i(x_i(t_i), \widehat{x}_{-i}(t_{-i}), t) dP(t_{-i}|t_i), \forall i, \forall t_i \in T_i, \forall x_i \in X_i. \end{aligned}$$

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- NE translation version under Bayesian update: Each player must be playing a best response to the conditional distribution of his opponents' strategies for each type that he might end up having. The integration can be interpreted as summation if types are finite and point mass probabilities apply.

Strategy Proof Equilibrium

- Wilson Doctrine (1987): A good game theory should not rely too heavily on assumptions of common knowledge.
  It is deficient to the extent it assumes other features to be common knowledge, such as one agent's probability assessment about another's preferences or information. Only by repeated weakening of common knowledge assumptions will the theory approximate reality.

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- In Chung and Ely (2007), the usual argument for imposing stronger solution concepts is that the resulting equilibrium will then be **detail free**: the rules would not have to be tailored to any fine details of the environment in which it is employed. Indeed, detail-freeness is the usual interpretation of the Wilson Doctrine. Thus, one may be interested in more robust and practical solution concepts that are free of prior or Bayesian beliefs.

- Given an n player strategic form game with incomplete information $H = (P, T_i, X_i, u_i)_{i=1}^n$ with interdependent value. A strategy profile $\widehat{x}$ is (pure strategy) **Ex Post Equilibrium (EPE)** if

  $$u_i(\widehat{x}_i(t_i), \widehat{x}_{-i}(t_{-i}), t) \geq u_i(x_i(t_i), \widehat{x}_{-i}(t_{-i}), t), \forall i, \forall t \in T, \forall x_i \in X_i.$$

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- Given an n player strategic form game with incomplete information $H = (P, T_i, X_i, u_i)_{i=1}^n$ with private value. A strategy profile $\widehat{x}$ is (pure strategy) **Dominant Strategy Equilibrium (DSE)** if

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- EPE and DSE are also called **Strategy Proof Equilibrium.** They have remarkable applications in mechanism design theory.

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- Generally, games with incomplete information are more complicated to compute than games with complete information. Prisoners' dilemma may have different outcomes with incomplete information. Cf Example 8.E.1and 2 in MWG.
- All auctions are games with incomplete information. All mechanism design problems are related to games with incomplete information. We will intensively analyze BNE, EPE, and DSE there.