Game Theory and Economics of Contracts
Lecture 4
Basics in Game Theory (2)

Yu (Larry) Chen

School of Economics, Nanjing University

Fall 2015

Extensive Form Game

- It uses game tree to represent the games. More clear for capturing the possible orders of events and therefore sequential games.

Extensive Form Game

- It uses game tree to represent the games. More clear for capturing the possible orders of events and therefore sequential games.
- The extensive form game contains the following information:

Extensive Form Game

- It uses game tree to represent the games. More clear for capturing the possible orders of events and therefore sequential games.
- The extensive form game contains the following information:
  1. the set of players

Extensive Form Game

- It uses game tree to represent the games. More clear for capturing the possible orders of events and therefore sequential games.
- The extensive form game contains the following information:
  1. the set of players
  2. the order of moves

Extensive Form Game

- It uses game tree to represent the games. More clear for capturing the possible orders of events and therefore sequential games.
- The extensive form game contains the following information:
    1. the set of players
    2. the order of moves
    3. the players' payoffs as a function of the moves that were made

Extensive Form Game

- It uses game tree to represent the games. More clear for capturing the possible orders of events and therefore sequential games.
- The extensive form game contains the following information:
  1. the set of players
  2. the order of moves
  3. the players' payoffs as a function of the moves that were made
  4. what the players' choices are when they move

Extensive Form Game

- It uses game tree to represent the games. More clear for capturing the possible orders of events and therefore sequential games.
- The extensive form game contains the following information:
  1. the set of players
  2. the order of moves
  3. the players' payoffs as a function of the moves that were made
  4. what the players' choices are when they move
  5. what each player knows when he makes his choices

Extensive Form Game

- It uses game tree to represent the games. More clear for capturing the possible orders of events and therefore sequential games.
- The extensive form game contains the following information:
  1. the set of players
  2. the order of moves
  3. the players' payoffs as a function of the moves that were made
  4. what the players' choices are when they move
  5. what each player knows when he makes his choices
  6. the probability distributions over any exogenous events.

Extensive Form Game

- It uses game tree to represent the games. More clear for capturing the possible orders of events and therefore sequential games.
- The extensive form game contains the following information:
  1. the set of players
  2. the order of moves
  3. the players' payoffs as a function of the moves that were made
  4. what the players' choices are when they move
  5. what each player knows when he makes his choices
  6. the probability distributions over any exogenous events.
- There is perfect information in such a game if each player, when making any decision, is perfectly informed of all the events that have previously occurred.

An n-player Extensive Game with Perfect Information

4-tuple (primitives), $G = (N, H, P, (u_i)_{i=1}^{n})$. Again they are common knowledge.

- A set N is the set of players.

An n-player Extensive Game with Perfect Information

4-tuple (primitives), $G = (N, H, P, (u_i)_{i=1}^n)$. Again they are common knowledge.

- A set N is the set of players.
- A set H of sequences (finite or infinite) is the **history**, and each component of a history is an action taken by a player, if H satisfies the following three properties.
  (1) The empty sequence $\varnothing$ is a member of H.
  (2) If $(a^k)_{k=1,\ldots,K} \in H$ (where K may be infinite) and $L < K$ then $(a^k)_{k=1,\ldots,L} \in H$.
  (3) If an infinite sequence $(a^k)_{k=1}^{\infty}$ satisfies $(a^k)_{k=1,\ldots,L} \in H$ for every positive integer L then $(a^k)_{k=1}^{\infty} \in H$.
  A history $h = (a^k)_{k=1,\ldots,K} \in H$ is terminal if it is infinite or if there is no a_{K+1} such that $(a^k)_{k=1,\ldots,K+1} \in H$. The set of terminal histories is denoted Z.

An n-player Extensive Game with Perfect Information

4-tuple (primitives), $G = (N, H, P, (u_i)_{i=1}^{n})$.

- A function P that assigns to each nonterminal history (each member of $H \backslash Z$) a member of N. (P is the player function, $P(h)$ being the player who takes an action after the history h.)

An n-player Extensive Game with Perfect Information

4-tuple (primitives), $G = (N, H, P, (u_i)_{i=1}^n)$.

- A function P that assigns to each nonterminal history (each member of $H \backslash Z$) a member of N. (P is the player function, $P(h)$ being the player who takes an action after the history h.)
- For each player $i \in N$, $u_i : Z \longrightarrow \mathbb{R}$ is the payoff function for player i.

Strategies

- A strategy of a player in an extensive game is a plan that specifies the action chosen by the player for every history after which it is his turn to move.

Strategies

- A strategy of a player in an extensive game is a plan that specifies the action chosen by the player for every history after which it is his turn to move.
- A n player **extensive form game with perfect information** $\Gamma = (N, H, P, (u_i)_{i=1}^n)$ implicitly determine the strategies of all players.

Strategies

- A strategy of a player in an extensive game is a plan that specifies the action chosen by the player for every history after which it is his turn to move.
- A n player **extensive form game with perfect information** $\Gamma = (N, H, P, (u_i)_{i=1}^n)$ implicitly determine the strategies of all players.
- Let h be a history of length k; we denote by (h, a) the history of length $k+1$ consisting of h followed by action a.

Strategies

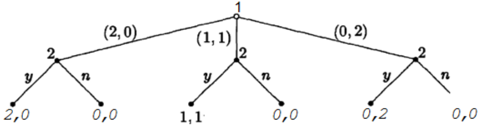
- A strategy of a player in an extensive game is a plan that specifies the action chosen by the player for every history after which it is his turn to move.
- A n player **extensive form game with perfect information** $\Gamma = (N, H, P, (u_i)_{i=1}^n)$ implicitly determine the strategies of all players.
- Let h be a history of length k; we denote by (h, a) the history of length $k + 1$ consisting of h followed by action a.
- After any nonterminal history h player $P(h)$ chooses an action from the set $A(h) = \{a : (h, a) \in H\}$.

Strategies

- A strategy of a player in an extensive game is a plan that specifies the action chosen by the player for every history after which it is his turn to move.
- A n player **extensive form game with perfect information** $\Gamma = (N, H, P, (u_i)_{i=1}^n)$ implicitly determine the strategies of all players.
- Let h be a history of length k; we denote by (h, a) the history of length $k+1$ consisting of h followed by action a.
- After any nonterminal history h player $P(h)$ chooses an action from the set $A(h) = \{a : (h, a) \in H\}$.
- s_i is the strategy of player i that assigns an action in $A(h)$ to each nonterminal history $h \in H \setminus Z$ for which $P(h) = i$

Example 1

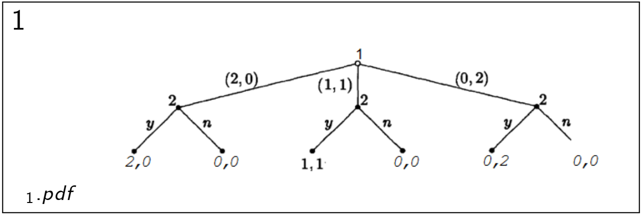
1



1.pdf

- Two people intend to share two desirable identical indivisible objects. One of them proposes an allocation, which the other then either accepts or rejects. In the event of rejection, neither person receives either of the objects. Each person cares only about the number of objects he obtains.

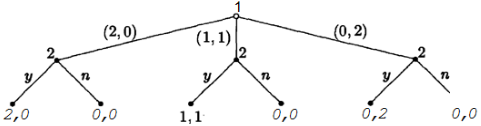
Example 1



- Two people intend to share two desirable identical indivisible objects. One of them proposes an allocation, which the other then either accepts or rejects. In the event of rejection, neither person receives either of the objects. Each person cares only about the number of objects he obtains.
- An extensive game that models this procedure is $(N, H, P, (u_i)_{i=1}^n)$ where $N = \{I, 2\}$; H consists of the ten histories $\varnothing$, $(2,0)$, $(1,1)$, $(0,2)$, $((2,0), y)$, $((2,0), n)$, $((1,1), y)$, $((1,1), n)$, $((0,2), y)$, $((0,2), n)$; $P(0) = 1$ and $P(h) = 2$ for every nonterminal history $h \neq \varnothing$.

Example 1

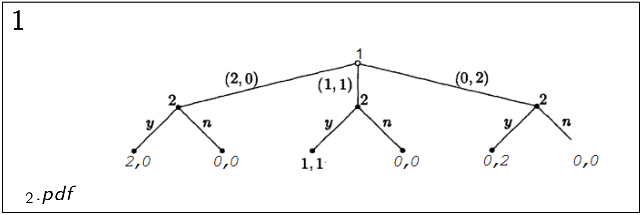
1



2.pdf

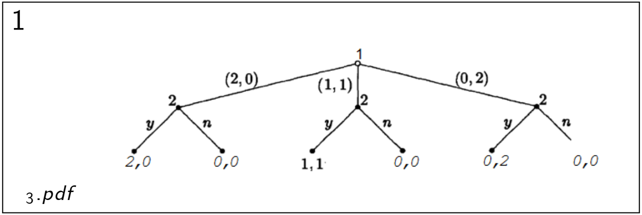
- $Z = \{((2,0),y),((2,0),n),((1,1),y),((1,1),n),((0,2),y),((0,2),n)\}$. u_i is defined by the ith number at every terminal node.

Example 1



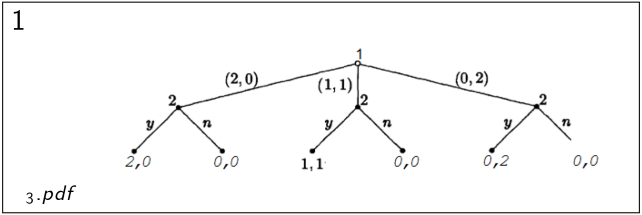
- $Z = \{((2,0),y),((2,0),n),((1,1),y),((1,1),n),((0,2),y),((0,2),n)\}$. u_i is defined by the ith number at every terminal node.
- Player 1 takes an action only after the initial history 0, so that we can identify each of her strategies with one of the three possible actions that she can take after this history: $(2,0)$, $(1,1)$, and $(0,2)$; Player 2 takes an action, either y or n, after each of the three histories $(2,0)$, $(1,1)$,and $(0,2)$. Thus we can identify each of his strategies by specifying either y or n to be the action that he chooses after the histories. His strategies represent a contingency plan (on the history he faces when he makes a choice)

An equivalent representation



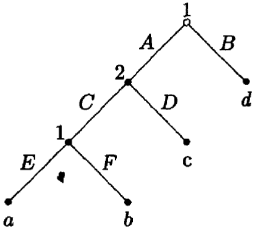
- This suggests an alternative definition of an extensive game in which the basic component is a tree (a connected graph with no cycles).

An equivalent representation



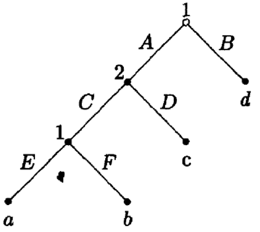
- This suggests an alternative definition of an extensive game in which the basic component is a tree (a connected graph with no cycles).
- In this formulation, each node corresponds to a history, and any pair of nodes that are connected corresponds to an action; the names of the actions are not part of the definition.

Example 2



- A strategy specifies the action chosen by a player for every history after which it is his turn to move, even for histories that, if the strategy is followed, are never reached.

Example 2



- A strategy specifies the action chosen by a player for every history after which it is his turn to move, even for histories that, if the strategy is followed, are never reached.
- In this game player 1 has four strategies AE, AF, BE, and BF. That is, her strategy specifies an action after the history (A, C) even if it specifies that she chooses B at the beginning of the game.

Nash Equilibrium

- A **Nash equilibrium** of an extensive game with perfect information $\Gamma = (N, H, P, (u_i)_{i=1}^n)$ is a strategy profile s^* such that for every player i we have

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$$

for every strategy s_i of player i.

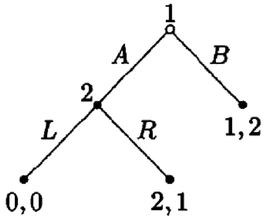
Nash Equilibrium

- A **Nash equilibrium** of an extensive game with perfect information $\Gamma = (N, H, P, (u_i)_{i=1}^{n})$ is a strategy profile s^* such that for every player i we have

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$$

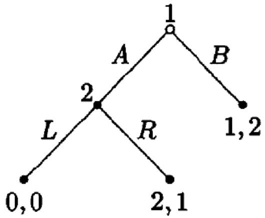
  for every strategy s_i of player i.
- (Zermelo 1913; Kuhn 1953) A finite game of perfect information has a pure-strategy NE.

Example 3



- 2 NE: (A, R) and (B, L), with payoff profiles (2,1) and (1,2). (B, L) is a NE because given that player 2 chooses L after the history A, it is optimal for player 1 to choose B at the start of the game (if she chooses A instead, then given player 2's choice she obtains 0 rather than 1), and given player 1's choice of B it is optimal for player 2 to choose L.

Example 3



- 2 NE: (A, R) and (B, L), with payoff profiles (2,1) and (1,2). (B, L) is a NE because given that player 2 chooses L after the history A, it is optimal for player 1 to choose B at the start of the game (if she chooses A instead, then given player 2's choice she obtains 0 rather than 1), and given player 1's choice of B it is optimal for player 2 to choose L.
- But NE (B, L) in this game lacks plausibility. If the history A were to occur then player 2 would, it seems, choose R over L, since he obtains a higher payoff by doing so. The equilibrium (B, L) is sustained by the "threat" of player 2 to choose L if player 1 chooses A. This threat is not credible since player 2 has no way of committing himself to this choice.

Sequential Rationality and Subgames

- **Sequential rationality**. Backward induction.
  The notion of equilibrium more sensible in extensive form games requires that the action prescribed by each player's strategy be optimal, given the other players' strategies, after **every history**.

Sequential Rationality and Subgames

- **Sequential rationality**. Backward induction.
  The notion of equilibrium more sensible in extensive form games requires that the action prescribed by each player's strategy be optimal, given the other players' strategies, after **every history**.
- The **subgame** of the extensive game with perfect information $\Gamma = (N, H, P, (u_i)_{i=1}^n)$ that follows the history h is the extensive game $\Gamma(h) = (N, H|_h, P|_h, (u_i|_h)_{i=1}^n)$ where $H|_h$ is the set of sequences h' of actions for which $(h, h') \in H$, $P|_h$ is defined by $P|_h(h') = P(h, h')$ for each $h' \in H|_h$, and $u_i|_h$ is defined by $u_i|_h(h') \geq u_i|_h(h'')$ if and only if $u_i(h, h') \geq u_i(h, h'')$, for each $h', h'' \in H|_h$.

Sequential Rationality and Subgames

- **Sequential rationality**. Backward induction.
  The notion of equilibrium more sensible in extensive form games requires that the action prescribed by each player's strategy be optimal, given the other players' strategies, after **every history**.
- The **subgame** of the extensive game with perfect information $\Gamma = (N, H, P, (u_i)_{i=1}^n)$ that follows the history h is the extensive game $\Gamma(h) = (N, H|_h, P|_h, (u_i|_h)_{i=1}^n)$ where $H|_h$ is the set of sequences h' of actions for which $(h, h') \in H$, $P|_h$ is defined by $P|_h(h') = P(h, h')$ for each $h' \in H|_h$, and $u_i|_h$ is defined by $u_i|_h(h') \geq u_i|_h(h'')$ if and only if $u_i(h, h') \geq u_i(h, h'')$, for each $h', h'' \in H|_h$.
- Given a strategy s_i of player i and a history h in the extensive game Γ, denote by $s_i|_h$ the strategy that s_i induces in the subgame $\Gamma(h)$ (i.e., $s_i|_h(h') = s_i(h, h')$ for each $h' \in H|_h$).

Subgame Perfect Equilibrium

- Given n player extensive form game with perfect information $\Gamma = (N, H, P, (u_i)_{i=1}^n)$, a **subgame perfect (Nash) equilibrium (SPE)** is a strategy profile s^* such that for every player i and every nonterminal history $h \in H \setminus Z$ for which $P(h) = i$,

$$u_i|_h(s_i^*|_h, s_{-i}^*|_h) \geq u_i|_h(s_i|_h, s_{-i}^*|_h)$$

for every strategy $s_i|_h$ of player i in the subgame $\Gamma(h)$.

Subgame Perfect Equilibrium

- Given n player extensive form game with perfect information $\Gamma = (N, H, P, (u_i)_{i=1}^n)$, a **subgame perfect (Nash) equilibrium (SPE)** is a strategy profile s^* such that for every player i and every nonterminal history $h \in H \setminus Z$ for which $P(h) = i$,

$$u_i|_h(s_i^*|_h, s_{-i}^*|_h) \geq u_i|_h(s_i|_h, s_{-i}^*|_h)$$

  for every strategy $s_i|_h$ of player i in the subgame $\Gamma(h)$.
- Equivalently, we can define a subgame perfect equilibrium to be a strategy profile s^* in Γ for which for any history h the strategy profile $s^*|_h$ is a Nash equilibrium of the subgame $\Gamma(h)$.

Subgame Perfect Equilibrium

- Given n player extensive form game with perfect information $\Gamma = (N, H, P, (u_i)_{i=1}^n)$, a **subgame perfect (Nash) equilibrium (SPE)** is a strategy profile s^* such that for every player i and every nonterminal history $h \in H \backslash Z$ for which $P(h) = i$,

$$u_i|_h(s_i^*|_h, s_{-i}^*|_h) \geq u_i|_h(s_i|_h, s_{-i}^*|_h)$$

  for every strategy $s_i|_h$ of player i in the subgame $\Gamma(h)$.
- Equivalently, we can define a subgame perfect equilibrium to be a strategy profile s^* in Γ for which for any history h the strategy profile $s^*|_h$ is a Nash equilibrium of the subgame $\Gamma(h)$.
- The notion of SPE eliminates NE in which the players' threats are not credible. Eg., in Example 3 the only SPE is (A, R) why?

Existence of SPE

- (Kuhn's Theorem) Every finite extensive game with perfect information has a subgame perfect equilibrium.

Existence of SPE

- (Kuhn's Theorem) Every finite extensive game with perfect information has a subgame perfect equilibrium.
- The proof (cf. Osborne and Rubinstein's book) is constructive: for each of the longest nonterminal histories, in the game we choose an optimal action for the player whose turn it is to move and replace each of these histories with a terminal history in which the payoff profile is that which results when the optimal action is chosen; then we repeat the procedure until going back to the start of the game.

Existence of SPE

- (Kuhn's Theorem) Every finite extensive game with perfect information has a subgame perfect equilibrium.
- The proof (cf. Osborne and Rubinstein's book) is constructive: for each of the longest nonterminal histories, in the game we choose an optimal action for the player whose turn it is to move and replace each of these histories with a terminal history in which the payoff profile is that which results when the optimal action is chosen; then we repeat the procedure until going back to the start of the game.
- The procedure (algorithm) used in this proof is often referred to as **backwards induction**. In addition to being a means by which to prove the theorem, this procedure is an algorithm for calculating the set of subgame perfect equilibria of a finite game.

Example 4: Stackelberg games

- A Stackelberg game is a two-player extensive game with perfect information in which a "leader" chooses an action from a set A_1 and a "follower," informed of the leader's choice, chooses an action from a set A_1. The solution usually applied to such games in economics is that of SPE.

Example 4: Stackelberg games

- A Stackelberg game is a two-player extensive game with perfect information in which a "leader" chooses an action from a set A_1 and a "follower," informed of the leader's choice, chooses an action from a set A_1. The solution usually applied to such games in economics is that of SPE.
- Some (but not all) subgame perfect equilibria of a Stackelberg game correspond to solutions of the maximization problem

$$\begin{aligned} & \max_{(a_1,a_2)\in A_1\times A_2} U_1(a_1,a_2) \\ s.t. a_2 \quad \in \quad & \arg\max_{a_2'\in A_2} U_2(a_1,a_2') \end{aligned}$$

where U_i is a payoff function that represents player i's preferences. If the set A_i of actions of each player i is compact and the payoff functions. U_i are continuous then this maximization problem has a solution.

Example 4: Stackelberg games

- A Stackelberg game is a two-player extensive game with perfect information in which a "leader" chooses an action from a set A_1 and a "follower," informed of the leader's choice, chooses an action from a set A_1. The solution usually applied to such games in economics is that of SPE.
- Some (but not all) subgame perfect equilibria of a Stackelberg game correspond to solutions of the maximization problem

$$\max_{(a_1,a_2)\in A_1\times A_2} U_1(a_1,a_2)$$

$$s.t. a_2 \in \arg\max_{a_2'\in A_2} U_2(a_1,a_2')$$

  where U_i is a payoff function that represents player i's preferences. If the set A_i of actions of each player i is compact and the payoff functions. U_i are continuous then this maximization problem has a solution.
- A concrete example.

Two Extensions of an extensive form Game: Exogenous Uncertainty

Sometimes we need to model situations in which there is some exogenous uncertainty.

An extensive game with perfect information and chance moves is a tuple $(N, H, P, f_c, (u_i)_{i=1}^{n})$ where, as before, N is a finite set of players and H is a set of histories, and

- P is a function from the nonterminal histories in H to $N \cup \{c\}$.(If $P(h) = c$ then **chance** determines the action taken after the history h.

Two Extensions of an extensive form Game: Exogenous Uncertainty

Sometimes we need to model situations in which there is some exogenous uncertainty.

An extensive game with perfect information and chance moves is a tuple $(N, H, P, f_c, (u_i)_{i=1}^n)$ where, as before, N is a finite set of players and H is a set of histories, and

- P is a function from the nonterminal histories in H to $N \cup \{c\}$.(If $P(h) = c$ then **chance** determines the action taken after the history h.
- For each $h \in H$ with $P(h) = c$, $f_c(\cdot|h)$ is a probability measure on $A(h)$; each such probability measure is assumed to be independent of every other such measure. ($f_c(a|h)$ is the probability that a occurs after the history h.)

Two Extensions of an extensive form Game: Exogenous Uncertainty

Sometimes we need to model situations in which there is some exogenous uncertainty.
An extensive game with perfect information and chance moves is a tuple $(N, H, P, f_c, (u_i)_{i=1}^n)$ where, as before, N is a finite set of players and H is a set of histories, and

- P is a function from the nonterminal histories in H to $N \cup \{c\}$.(If $P(h) = c$ then **chance** determines the action taken after the history h.
- For each $h \in H$ with $P(h) = c$, $f_c(\cdot|h)$ is a probability measure on $A(h)$; each such probability measure is assumed to be independent of every other such measure. ($f_c(a|h)$ is the probability that a occurs after the history h.)
- For each player i, u_i is a (expected) payoff function on *lotteries* over the set of terminal histories (with respect to f_c).

Two Extensions of an extensive form Game: Exogenous Uncertainty

- A strategy for each player i is defined as before.

Two Extensions of an extensive form Game: Exogenous Uncertainty

- A strategy for each player i is defined as before.
- The outcome of a strategy profile is a probability distribution over terminal histories.

Two Extensions of an extensive form Game: Exogenous Uncertainty

- A strategy for each player i is defined as before.
- The outcome of a strategy profile is a probability distribution over terminal histories.
- The definition of a subgame perfect equilibrium is the same as before.

Two Extensions of an extensive form Game: Simultaneous Moves

Sometimes we need to model situations in which players move simultaneously after certain histories, each of them being fully informed of all past events when making his choice. (1 principal contracting with many agents)
An **extensive game with perfect information and simultaneous moves (**also, **multi-stage game)** is a tuple $(N, H, P, (u_i)_{i=1}^n)$ where N, H, and u_i for each i are the same as before, P is a function that assigns to each nonterminal history *a set of players*, and H and P jointly satisfy the condition that for every nonterminal history h there is a collection $\{A_i(h)\}_{i \in P(h)}$ of sets for which $A(h) = \{a : (h, a) \in H\} = \times_{i \in P(h)} A_i(h)$.

Two Extensions of an extensive form Game: Simultaneous Moves

Besides,

- a history in such a game is a sequence of vectors; the components of each vector a^k are the actions taken by the players whose turn it is to move after the history $(a^l)_{l=1}^{k-1}$. The set of actions among which each player $i \in P(h)$ can choose after the history h is $A_i(h)$; the interpretation is that the choices of the players in $P(h)$ are made simultaneously;

Two Extensions of an extensive form Game: Simultaneous Moves

Besides,

- a history in such a game is a sequence of vectors; the components of each vector a^k are the actions taken by the players whose turn it is to move after the history $(a^l)_{l=1}^{k-1}$. The set of actions among which each player $i \in P(h)$ can choose after the history h is $A_i(h)$; the interpretation is that the choices of the players in $P(h)$ are made simultaneously;
- a strategy of player i in such a game is a function that assigns an action in $A_i(h)$ to every nonterminal history h for which $i \in P(h)$. The definition of a subgame perfect equilibrium is the same as before with the exception that "$P(h) = i$" is replaced by "$i \in P(h)$".

Extensive game with Imperfect Information

- If a player is not able to observe all her opponents' previous moves when it is her turn to move, the game has imperfect information. Things will be a lot more complicated.

Extensive game with Imperfect Information

- If a player is not able to observe all her opponents' previous moves when it is her turn to move, the game has imperfect information. Things will be a lot more complicated.
- Relevant concepts:
  information set
  Belief Update
  Perfect Bayesian Equilibrium – sender-receiver games
  Sequential Equilibrium.

Cooperative Game: Bargaining

- When a buyer and a seller negotiate the price of a house they are faced with a **bargaining problem**. Similarly, two trading countries bargaining over the terms of trade, a basketball player discussing his contract with the owners of a team, or two corporations arguing over the details of a joint venture.

Cooperative Game: Bargaining

- When a buyer and a seller negotiate the price of a house they are faced with a **bargaining problem**. Similarly, two trading countries bargaining over the terms of trade, a basketball player discussing his contract with the owners of a team, or two corporations arguing over the details of a joint venture.
- In all these bargaining situations, there is usually a set S of **alternative outcomes** and the two sides have to agree on some element of this set. Once an agreement has been reached, the bargaining is over, and the two sides then receive their respective payoffs. In case they cannot agree, the result is usually the status-quo, and we say there is **disagreement**.

Cooperative Game: Bargaining

- When a buyer and a seller negotiate the price of a house they are faced with a **bargaining problem**. Similarly, two trading countries bargaining over the terms of trade, a basketball player discussing his contract with the owners of a team, or two corporations arguing over the details of a joint venture.
- In all these bargaining situations, there is usually a set S of **alternative outcomes** and the two sides have to agree on some element of this set. Once an agreement has been reached, the bargaining is over, and the two sides then receive their respective payoffs. In case they cannot agree, the result is usually the status-quo, and we say there is **disagreement**.
- It is quite clear that the two sides will not engage in bargaining, unless there are outcomes in S which give both sides a higher payoff than the payoffs they receive from the status-quo. Thus, if (d_1, d_2) are the payoffs from the **disagreement point**, then the interesting part of S consists of those outcomes which give both sides higher payoffs than the disagreement payoffs.

Bargaining Problem

- A **n-person bargaining problem (or game)** consists of n persons (or players) indexed by $i \in N$, a set S of *feasible alternatives* (or bargaining outcomes or simply outcomes), and a utility function u_i on S for each player i, such that

Bargaining Problem

- A **n-person bargaining problem (or game)** consists of n persons (or players) indexed by $i \in N$, a set S of *feasible alternatives* (or bargaining outcomes or simply outcomes), and a utility function u_i on S for each player i, such that
  1. $u_i(s) \geq d_i$ for every $s \in S$, where d_i is *the payoff from the disagreement point*, and

Bargaining Problem

- A **n-person bargaining problem (or game)** consists of n persons (or players) indexed by $i \in N$, a set S of *feasible alternatives* (or bargaining outcomes or simply outcomes), and a utility function u_i on S for each player i, such that
  1. $u_i(s) \geq d_i$ for every $s \in S$, where d_i is *the payoff from the disagreement point*, and
  2. at least for one $s \in S$ for which $u_i(s) > d_i$.

Bargaining Problem

- A **n-person bargaining problem (or game)** consists of n persons (or players) indexed by $i \in N$, a set S of *feasible alternatives* (or bargaining outcomes or simply outcomes), and a utility function u_i on S for each player i, such that
  1. $u_i(s) \geq d_i$ for every $s \in S$, where d_i is *the payoff from the disagreement point*, and
  2. at least for one $s \in S$ for which $u_i(s) > d_i$.
- Condition (2) guarantees that there is a feasible alternative which makes all players strictly better-off relative to the disagreement point. This condition makes the bargaining problem non-trivial.

Bargaining Problem

- A **n-person bargaining problem (or game)** consists of n persons (or players) indexed by $i \in N$, a set S of *feasible alternatives* (or bargaining outcomes or simply outcomes), and a utility function u_i on S for each player i, such that
  1. $u_i(s) \geq d_i$ for every $s \in S$, where d_i is *the payoff from the disagreement point*, and
  2. at least for one $s \in S$ for which $u_i(s) > d_i$.
- Condition (2) guarantees that there is a feasible alternative which makes all players strictly better-off relative to the disagreement point. This condition makes the bargaining problem non-trivial.
- Formally, we can write a bargaining problem as a triplet $\mathcal{B} = (S, (u_i, d_i)_{i=1}^{n})$, where S, u_1 and u_2 satisfy properties (1) and (2).

Bargaining Problem

- Now focus on **2-person bargaining problem** for simplicity. Everything can be easily generalized to n-person cases.

Bargaining Problem

- Now focus on **2-person bargaining problem** for simplicity. Everything can be easily generalized to n-person cases.
- We formally define **a solution for a bargaining problem** to be a rule (i.e., a set-valued function) that assigns to each bargaining game $\mathcal{B}$ a subset $s(\mathcal{B})$ of the set of its outcomes S.

Bargaining Problem

- Now focus on **2-person bargaining problem** for simplicity. Everything can be easily generalized to n-person cases.
- We formally define **a solution for a bargaining problem** to be a rule (i.e., a set-valued function) that assigns to each bargaining game $\mathcal{B}$ a subset $s(\mathcal{B})$ of the set of its outcomes S.
- We can think of the set $s(\mathcal{B})$ as the collection of all *mutually satisfactory agreements* of the bargaining game $\mathcal{B}$ or, simply, as the solutions of the bargaining game.

Bargaining Problem

- Now focus on **2-person bargaining problem** for simplicity. Everything can be easily generalized to n-person cases.
- We formally define **a solution for a bargaining problem** to be a rule (i.e., a set-valued function) that assigns to each bargaining game $\mathcal{B}$ a subset $s(\mathcal{B})$ of the set of its outcomes S.
- We can think of the set $s(\mathcal{B})$ as the collection of all *mutually satisfactory agreements* of the bargaining game $\mathcal{B}$ or, simply, as the solutions of the bargaining game.
- Obviously, any such rule should satisfy certain reasonable conditions.

Bargaining Problem

Obviously, any such rule should satisfy certain reasonable conditions.

- **Pareto Optimality or Efficiency**: An outcome $s^* \in S$ is said to be **Pareto optimal (or Pareto efficient)** if there is no other outcome $s \in S$ satisfying

Bargaining Problem

Obviously, any such rule should satisfy certain reasonable conditions.

- **Pareto Optimality or Efficiency**: An outcome $s^* \in S$ is said to be **Pareto optimal (or Pareto efficient)** if there is no other outcome $s \in S$ satisfying
  1. $u_1(s) \geq u_1(s^*)$ and $u_2(s) \geq u_2(s^*)$, and

Bargaining Problem

Obviously, any such rule should satisfy certain reasonable conditions.

- **Pareto Optimality or Efficiency**: An outcome $s^* \in S$ is said to be **Pareto optimal (or Pareto efficient)** if there is no other outcome $s \in S$ satisfying
  1. $u_1(s) \geq u_1(s^*)$ and $u_2(s) \geq u_2(s^*)$, and
  2. $u_i(s) > u_i(s^*)$ for at least one player i.

Bargaining Problem

Obviously, any such rule should satisfy certain reasonable conditions.

- **Pareto Optimality or Efficiency**: An outcome $s^* \in S$ is said to be **Pareto optimal (or Pareto efficient)** if there is no other outcome $s \in S$ satisfying
  1. $u_1(s) \geq u_1(s^*)$ and $u_2(s) \geq u_2(s^*)$, and
  2. $u_i(s) > u_i(s^*)$ for at least one player i.
- Intuition: This guarantees that there is no further possibility of strictly improving the utility of one of the players, while leaving the other at least as well off as she was before.

Bargaining Problem

Obviously, any such rule should satisfy certain reasonable conditions.

- **Pareto Optimality or Efficiency**: An outcome $s^* \in S$ is said to be **Pareto optimal (or Pareto efficient)** if there is no other outcome $s \in S$ satisfying
  1. $u_1(s) \geq u_1(s^*)$ and $u_2(s) \geq u_2(s^*)$, and
  2. $u_i(s) > u_i(s^*)$ for at least one player i.
- Intuition: This guarantees that there is no further possibility of strictly improving the utility of one of the players, while leaving the other at least as well off as she was before.
- A solution rule $s(\cdot)$ is said to be Pareto optimal if for every game $\mathcal{B}$ the set $s(\mathcal{B})$ consists of Pareto optimal outcomes.

Bargaining Problem

Another reasonable condition.

- **Independence of Irrelevant Alternatives**: A solution rule s is **independent of irrelevant alternatives** if for every bargaining game $\mathcal{B} = (S, (u_1, d_1), (u_2, d_2))$ and for every subset T of S satisfying $(d_1, d_2) \in \mathcal{U}_T = \{(u_1(s), u_2(s)) : s \in T\}$ and $s(\mathcal{B}) \subseteq T$, we have $s(\mathcal{B}_T) = s(\mathcal{B})$, where $\mathcal{B}_T$ is the bargaining game $\mathcal{B}_T = (T, (u_1, d_1), (u_2, d_2))$.

Bargaining Problem

Another reasonable condition.

- **Independence of Irrelevant Alternatives**: A solution rule s is **independent of irrelevant alternatives** if for every bargaining game $\mathcal{B} = (S, (u_1, d_1), (u_2, d_2))$ and for every subset T of S satisfying $(d_1, d_2) \in \mathcal{U}_T = \{(u_1(s), u_2(s)) : s \in T\}$ and $s(\mathcal{B}) \subseteq T$, we have $s(\mathcal{B}_T) = s(\mathcal{B})$, where $\mathcal{B}_T$ is the bargaining game $\mathcal{B}_T = (T, (u_1, d_1), (u_2, d_2))$.
- Intuition: any acceptable bargaining solution rule should remain acceptable if we throw away alternatives that have already been considered to be less desirable by both players.

Bargaining Problem

Last reasonable condition.

- **Independence of Linear Transformations**: A bargaining solution $s(\cdot)$ is said to be **independent of linear transformations** if for any bargaining game $\mathcal{B} = (S, (u_1, d_1), (u_2, d_2))$, and any linear utility functions of the form $v_i = a_i + b_i u_i$, where the a_i and b_i are constants with $b_i > 0$ for each i, the bargaining game $\mathcal{B}^* = (S, (v_1, b_1 d_1 + a_1), (v_2, b_2 d_2 + a_2))$ satisfies $s(\mathcal{B}^*) = s(\mathcal{B})$.

Bargaining Problem

Last reasonable condition.

- **Independence of Linear Transformations**: A bargaining solution $s(\cdot)$ is said to be **independent of linear transformations** if for any bargaining game $\mathcal{B} = (S, (u_1, d_1), (u_2, d_2))$, and any linear utility functions of the form $v_i = a_i + b_i u_i$, where the a_i and b_i are constants with $b_i > 0$ for each i, the bargaining game $\mathcal{B}^* = (S, (v_1, b_1 d_1 + a_1), (v_2, b_2 d_2 + a_2))$ satisfies $s(\mathcal{B}^*) = s(\mathcal{B})$.
- Intuition: This guarantees that the bargaining solution rule will not be affected by changing the scale or units in which we measure utility.

Bargaining Problem

- We now proceed to describe a solution rule which satisfies all these conditions.

Bargaining Problem

- We now proceed to describe a solution rule which satisfies all these conditions.
- We start by associating to each bargaining game $\mathcal{B} = (S, (u_1, d_1), (u_2, d_2))$ the function $g_{\mathcal{B}} : S \longrightarrow \mathbb{R}$ defined by

$$g_{\mathcal{B}}(s) = [u_1(s) - d_1][u_2(s) - d_2],$$

and let $\sigma(B)$ be the set of all maximizers of the function $g_{\mathcal{B}}$, i.e., $\sigma(\mathcal{B}) = \arg\max_{s \in S} g_{\mathcal{B}}(s)$.

Bargaining Problem

- We now proceed to describe a solution rule which satisfies all these conditions.
- We start by associating to each bargaining game $\mathcal{B} = (S, (u_1, d_1), (u_2, d_2))$ the function $g_{\mathcal{B}} : S \longrightarrow \mathbb{R}$ defined by

$$g_{\mathcal{B}}(s) = [u_1(s) - d_1][u_2(s) - d_2],$$

  and let $\sigma(B)$ be the set of all maximizers of the function $g_{\mathcal{B}}$, i.e., $\sigma(\mathcal{B}) = \arg\max_{s \in S} g_{\mathcal{B}}(s)$.
- We shall call $\sigma(\cdot)$ the **Nash solution rule** and for any bargaining game $\mathcal{B}$ we shall refer to the members (if there are any) of $\sigma(\mathcal{B})$ as the **Nash solutions** of $\mathcal{B}$.

Remarks

- Such a form of $g_{\mathcal{B}}(s)$ implicitly assumes there is symmetric bargaining power between players 1 and 2. To address the asymmetric bargaining power, one may simply consider $g_{\mathcal{B}}(s) = [u_1(s) - d_1]^{\alpha}[u_2(s) - d_2]^{\beta}$, where $\alpha, \beta \geq 0, \alpha + \beta = 1$, and α and β will denote the weights of respective bargaining power.

Remarks

- Such a form of $g_{\mathcal{B}}(s)$ implicitly assumes there is symmetric bargaining power between players 1 and 2. To address the asymmetric bargaining power, one may simply consider $g_{\mathcal{B}}(s) = [u_1(s) - d_1]^{\alpha}[u_2(s) - d_2]^{\beta}$, where $\alpha, \beta \geq 0, \alpha + \beta = 1$, and α and β will denote the weights of respective bargaining power.
- In such a bargaining game, the objective function to be maximized is not an individual payoff but a *joint (coalitional) payoff*. So it is a **cooperative (coalitional) game**.

Bargaining Problem

- Theorem (Nash 1950)
  On the class of bargaining games with compact sets of utility allocations $\mathcal{U} = \{(u_1(s), u_2(s)) : s \in S\}$, the Nash rule $\sigma(\cdot)$ is Pareto optimal, independent of irrelevant alternatives, and independent of linear transformations.

Bargaining Problem

- Theorem (Nash 1950)
  On the class of bargaining games with compact sets of utility allocations $\mathcal{U} = \{(u_1(s), u_2(s)) : s \in S\}$, the Nash rule $\sigma(\cdot)$ is Pareto optimal, independent of irrelevant alternatives, and independent of linear transformations.
- Example 5: Suppose two individuals are bargaining over a sum of money; say \$100. If they cannot agree on how to divide the money, none of them gets any money.