Game Theory and Economics of Contracts
Lecture 5
Static Single-agent Moral Hazard Model

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Principal-Agent Relationship

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- P normally has full bargaining power, so she need to design an **incentive scheme** to A(s) to motivate the informed agent(s) to behave in her best interests. —> Contracting!

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- Not surprisingly, P will want to influence A's actions. This influence will often take the form of a contract that has P compensating A contingent on either his actions or the consequences of his actions.
- But such actions are normally unobservable to P but observable to A himself, so they are also not contractible that is, P cannot make contracts directly contingent on the actions. Two parties have different objectives. A is then inclined to pick the action in his best interests but not in P's best interest. So actions by A will then impose a (negative) externality on P. — Moral hazard (from hidden actions of A)

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- But because this behavior is known as moral hazard, since it imposes an externality on the insurance company, insurance companies were eager to develop contracts that guarded against it.

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- Eg. many insurance contracts have *deductibles*—the first k dollars of damage must be paid by the insured rather than the insurance company. Because the insured now has $\$k$ at risk, he will think twice about parking in a bad neighborhood. That is, the insurance contract is designed to mitigate the externality that A—the insured—imposes on P—the insurance company.

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- Although principal-agent analysis is more general than this, the name "moral hazard" has stuck and, so, the types of problems considered here are often referred to as moral-hazard problems.

More Examples for MH Problem

Principal	Agent	Problem	Solution
Employer	Employee	Induce employee to take actions that increase employer's profits, but which he finds personally costly.	Base employee's compensation on employer's profits.
Plaintiff	Attorney	Induce attorney to expend costly effort to increase plaintiff's chances of prevailing at trial.	Make attorney's fee contingent on damages awarded plaintiff.
Homeowner	Contractor	Induce contractor to complete work (e.g., remodel kitchen) on time.	Give contractor bonus for completing job on time.
Landlord	Tenant	Induce tenant to make investments (e.g., in time or money) that preserve or enhance property's value to the landlord.	Pay the tenant a fraction of the increased value (e.g., share-cropping contract). Alternatively, make tenant post deposit to be forfeited if value declines too much.

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- A's action is hidden; that is, he knows what action he has taken but P does not directly observe his action. Moreover, A has complete discretion in choosing his action from some set of feasible actions.

Single Agency Moral Hazard Model Setting

- The actions determine, usually stochastically, some performance measures (or outcomes, signals). The contract is a function of (at least some) of these performance variables. In particular, the contract can be a function of the observable, **verifiable** performance measures. Information is verifiable if it can be observed perfectly (i.e., without error) by third parties, who might be called upon to adjudicate a dispute between P and A.

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- The structure of the situation is common knowledge between the players.

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- Many of these actions are unobservable by his company. The company can, however, measure in a verifiable way the number of orders or revenue he generates.
- Because these measures are, presumably, correlated with his actions (i.e., the harder he works, the more sales he generates on average), it may make sense for the company to base his pay on his sales—put him on commission—to induce him to expend the appropriate level of effort.

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- Once the contract has been agreed to, the only player to take further actions is A. The game is played once. In particular, there is only one period in which A takes actions and A completes his actions before any performance measures are realized.

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- x is observable and verifiable. So enforceable contracts can be written on the variable, x. The nature of P's contract offer will be a wage schedule, $w : \mathcal{X} \rightarrow \mathbb{R}$, according to which A is rewarded. P has residual claim rights, i.e., P can keep $x - w(x)$ as her profit.

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- We also assume that P has full commitment and will not alter the contract $w(x)$ later.

Standard Static P-A MH Model

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- But A's action results in a monetary disutility of $c(a)$, which is continuously differentiable, increasing and strictly convex. The monetary utility of P is $V(x - w(x))$, where $V' > 0 \geq V''$. A's net utility is separable in cost of effort and money: $U(w(x), a) \equiv u(w(x)) - c(a)$; where $u' > 0 \geq u''$.

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- Game Tree.

General Static P-A Problem with MH

- P solves the following program

$$\max_{w\in\mathcal{F};a\in\mathcal{A}} \int_{\mathcal{X}} V(x-w(x))P(dx|a)$$

$$s.t. \int_{\mathcal{X}} u(w(x))P(dx|a) - c(a) \geq r,$$

$$a \in \arg\max_{a'\in\mathcal{A}} \int_{\mathcal{X}} u(w(x))P(dx|a') - c(a'),$$

where the first constraint is A's participation or individual rationality (IR) constraint, and the second is A's incentive compatibility (IC) constraint. Any behavior is motivated by economic interests!

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- Interpretation. Why is a also the choice variable controlled by P?

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- $F_a(x|a) < 0, \forall x \in (\underline{x}, \overline{x})$,i.e., action produces a first-order stochastic dominant shift on X.
- Since our support is fixed, $F_a(\underline{x}|a) = F_a(\overline{x}|a) = 0$ for any a.

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- The Lagrangian is

$$\mathcal{L} = \int_{\underline{x}}^{\overline{x}} [V(x - w(x)) + \lambda u(w(x))] f(x|a) dx - \lambda (c(a) + r),$$

where is the Lagrangian multiplier associated with the IR constraint; it represents the shadow price of income to A in each state.

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- Assuming an interior solution, we have as first-order conditions after simplification,

$$\frac{V'(x - w(x))}{u'(w(x))} = \lambda, x \in \mathcal{X},$$

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  and the IR constraint is binding (Holmstrom 1979).
- The first condition is known as the **Borch rule**: the ratios of marginal utilities of income are equated across states under an optimal contract. Note that it holds for every x and not just in expectation. The second condition is the choice of effort condition.

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- The IC constraint implies (assuming an interior optimum) that

$$\int_{\underline{x}}^{\overline{x}} u(w(x)) fa(x; a) dx - c'(a) = 0;$$

$$\int_{\underline{x}}^{\overline{x}} u(w(x)) fa(x; a) dx - c'(a) \leq 0;$$

which are the local first- and second-order conditions for a maximum.

First-Order Approach (FOA) to Incentives Contracts

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- Using μ as the multiplier on the first-order condition w.r.t IC, the Lagrangian of the FOA program is

$$\begin{aligned}\mathcal{L} &= \int_{\underline{x}}^{\overline{x}} [V(x - w(x)) + \lambda u(w(x))] f(x|a) dx - \lambda(c(a) + r) \\ &+ \mu(\int_{\underline{x}}^{\overline{x}} u(w(x)) fa(x; a) dx - c'(a)).\end{aligned}$$

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- **Modified Borch rule**: the marginal rates of substitution may vary if $\mu > 0$ to consider the incentives effect of $w(x)$. Thus, risk-sharing will generally be inefficient, compared with FB.

First-Order Approach (FOA) to Incentives Contracts

- Consider a simple two-action (finite) case in which P wishes to induce the high action: $\mathcal{A} = \{a_L; a_H\}$. Then the IC constraint implies the inequality

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- In both cases, providing $\mu > 0$, A is rewarded for outcomes which have higher relative frequency under high action.
- Theorem 1 (Holmstrom, 1979) Assume that the FOA program is valid. Then at the optimum of the FOA program, $\mu > 0$. The proof of the theorem relies upon first-order stochastic dominance $F_a(x|a) < 0$ and risk aversion $u'' < 0$.

Monotonicity of the Optimal Contracts

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- Theorem (Holmstrom, 1979). Under FOA, if F satisfies the MLRP, then the wage contract $w(x)$ is increasing in x.

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$$f(x, s|a) = y(x|a)z(x|s)$$

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- Theorem 3 (Holmstrom, [1979], Shavell [1979]). Assume that the FOA program is valid and yields $w(x)$ as a solution. Then there exists a new contract, $w(x, s)$, that strictly Pareto dominates $w(x)$ iff s is informative about $a \in \mathcal{A}$.

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- Proof: Using the FOA program, but allowing $w(\cdot)$ to depend upon s as well as x, the first-order condition determining w is given by

$$\frac{V'(x - w(x,s))}{u'(w(x,s))} = \lambda + \mu \frac{f_a(x,s|a)}{f(x,s|a)}, \forall x \in \mathcal{X}.$$

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- Implication: P can restrict attention to wage contracts that depend only upon a set of sufficient statistics for A's action, since we normally assume there is no cost of monitoring one more signal.
- Moreover, any informative signal about A's action should be included in the optimal contract! (But if monitoring cost is considerable, still no!)

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- A distribution satisfies the Convexity of Distribution Function Condition (CDFC) iff

$$F(x, \gamma a + (1-\gamma)a') \leq \gamma F(x;a) + (1-\gamma)F(x;a);$$

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- Theorem (Rogerson, 1985) The first-order approach is valid if $F(x|a)$ satisfies the MLRP and CDFC conditions.
- Therefore, MLRP and CDFC guarantees the FOA program is valid and yields a monotonic wage schedule.

A Natural Case: Linear Contracts with Normally Distributed Performance and Exponential Utility

- Performance $x = a + \varepsilon$, where $\varepsilon \sim N(0, \sigma^2)$. P is risk neutral, while A has a utility function:

$$u(w; a) = -e^{-\gamma(w - c(a))},$$

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- Transform the original problem to a certainty equivalent version $\widehat{w}(a) = w - c(a)$.